INSTRUMENTAL VARIABLE ESTIMATION OF NONPARAMETRIC MODELS

BY WHITNEY K. NEWEY AND JAMES L. POWELL

Presented by Jiyeon Seo and Yonghyeon Yang

22nd, Feb.

- Model

  - Structural Form

  : $y = g(x, z_1, \varepsilon)$, $E[\varepsilon|z] = 0$, $z = (z_1, z_2)$

  - Objective : To identify $g(\cdot)$.

  - If g is invertible, $\varepsilon = \rho(y, x, z_1)$

- Additive Model

$$y = g_0(x, z_1) + \varepsilon \ , \quad E[\varepsilon|z] = 0, \quad z = (z_1, z_2)$$

$$\pi(z) = E[y|z] = E[g_0(x, z_1)|z]$$

$$= \int g_0(x, z_1)\, dF(x|z)$$

: Nonparametric version of Reduced Form for y

- Identification

: π and F are obtained from data

: Identification Problem
= Problem of obtaining the unique solution of g from the reduced form

- **Proposition 2.1**

: Define $\delta(x, z_1) \equiv \tilde{g}(x, z_1) - g_0(x, z_1)$.
: g_0 is identified if only if
. $\forall \delta(x, z_1)$ with finite expectation, $E[\delta(x, z_1) | z] = 0$ implies $\delta(x, z_1) = 0$.

- completeness of conditional distribution of x given z
- (cf. complete sufficient statistic)
- special case?

- ESTIMATION: NP2SLS

: $\widehat{\pi}(z) = \int g(x, z_1)\, d\widehat{F}(x|z)$
: i.i.d. data (y_i, x_i, z_i)

- Issue : "Ill-posed inversion" problem

- How to overcome?

CASE1 : $w = (x, z_1)$ is bounded.

$$g_0(w) \simeq \sum \gamma_j p_j(w)$$

$$\pi(z) \simeq \int \sum \gamma_j p_j(w) dF(x|z) = \sum \gamma_j E\left[p_j(w)|z\right]$$

- Example

$$
\begin{aligned}
g_0(w) &= w\gamma \\
y &= w\gamma + \varepsilon \\
w &= z\alpha + \eta \quad where \;\; z \perp \eta, \; z \perp \varepsilon \\
P_z y &= P_z w\gamma + P_z\varepsilon, \qquad E\left[P_z\varepsilon | z\right] = 0 \\
\gamma &= \left(\left(P_z w\right)' P_z w\right)^{-1} \left(P_z w\right)' P_z y \qquad : 2SLS
\end{aligned}
$$

CASE2 : w is unbounded.

$$g_0(w) = a(w)'\beta + g_1(w)$$

- $g_0(w) = a(w)'\beta + \sum_j \gamma_j p_j(\widehat{w}) \quad where \quad \widehat{w} = \widehat{\Sigma}_1^{-1/2}(w - \widehat{\mu}_1)$

- $\pi(z) = \int g_0(w)\,dF(x|z) = E[a(w)|z]'\beta + \sum_j \gamma_j E\left[p_j(\widehat{w})|z\right]$

- Let $y = \pi(z) + \varepsilon$

$$\min \frac{1}{n}\sum_i \widehat{\varepsilon}^2 = \frac{1}{n}\sum_i \left\{y_i - \widehat{E}[a|z_i]'\beta - \sum_j \gamma_j \widehat{E}\left[p_j(\widehat{w})|z_i\right]\right\}^2$$

$$s.t. \quad \beta'\beta \leq B_\beta, \quad ||g_1|| \leq B_1$$

$$\begin{pmatrix} \widehat{\beta} \\ \widehat{\gamma} \end{pmatrix} = \left(\widehat{R}'\widehat{R} + \begin{pmatrix} \widehat{\zeta}_\beta I & 0 \\ 0 & \widehat{\zeta}_g \Lambda_J \end{pmatrix}\right)^{-1} \widehat{R}'Y$$

$$where \ \widehat{R} = \begin{pmatrix} \widehat{E}[a|z_1]' & \widehat{E}[p_1|z_1] & \cdots & \widehat{E}[p_J|z_1] \\ \vdots & & & \vdots \\ \widehat{E}[a|z_n]' & \widehat{E}[p_1|z_n] & \cdots & \widehat{E}[p_J|z_n] \end{pmatrix}$$

- Extension : General CMR

$$
\begin{aligned}
\widehat{\rho}\left(z_i;\theta\right) &= \left[\sum\nolimits_l \rho\left(y_l, x_l;\theta\right) q_l^{K\prime}/n\right] \widehat{M}^{-} q^K\left(z_i\right) \\
\widehat{\varepsilon} &= \widehat{\rho} \\
&= Q^K\left(Q^{K\prime}Q^K\right)^{-1} Q^{K\prime}\rho \qquad \textit{where} \quad Q^K = \left(q_1^K, \cdots, q_n^K\right)' \\
\widehat{\theta} &= \arg\min_{\theta\in\Theta_J} \sum\nolimits_i \widehat{\rho}\left(z_i;\theta\right)' \widehat{A}\widehat{\rho}\left(z_i;\theta\right) \qquad \textit{subject to constraints}
\end{aligned}
$$

• CONSISTENCY : General CMR

Assumption 1 : $\theta_0 \in \Theta$ is the only $\theta \in \Theta$ satisfying $E\left[\rho\left(y,x,\theta\right)|z\right]=0$

Assumption 2 : For any $b\left(z\right)$ with $E\left[b\left(z\right)^2\right]<\infty$, there is π_K with $E\left[\left\{b\left(z\right)-q^K\left(z\right)'\pi_K\right\}^2\right]\rightarrow 0,\ K\rightarrow\infty$, and $K/n\rightarrow 0$. Also, $\widehat{A}\xrightarrow{p} A$, and A is positive definite and constant.

Assumption 3 : $E\left[||\rho\left(y,x,\theta_0\right)||^2|z\right]$ is bounded and there exists $M\left(y,x\right),\ \nu>0$ such that for all $\tilde{\theta},\ \theta\in\Theta,\ ||\rho\left(y,x,\tilde{\theta}\right)-\rho\left(y,x,\theta\right)||\leq M\left(y,x\right)||\tilde{\theta}-\theta||^{\nu}$ and $E\left[M\left(y,x\right)^2|z\right]$ is bounded.

Assumption 4 : $\theta_0\in\Theta$, and Θ is compact for the norm $||\theta||$.

Assumption 5 :
For any $\theta\in\Theta$ there exists $\theta_J\in\Theta_J$ such that $\lim_{J\rightarrow\infty}||\theta_J-\theta||=0$.

Theorem 4.1 :
If Assumptions 1-5 are satisfied and $J\rightarrow\infty$, then $||\widehat{\theta}-\theta_0||\xrightarrow{p} 0$.