Econometrics Exercise 1

Consider Random Effects (Maddala) model.

$$y_{it} = x_{it}'\beta + v_i + w_{it}$$

where

$$\begin{aligned}
&v_i \text{ and } w_{it} \text{ are independent random variable.}\\
E[v_i] &= 0\\
E[w_{it}] &= 0\\
Var(v_i) &= \sigma_v^2\\
Var(w_{it}) &= \sigma_w^2
\end{aligned}$$

Write this as

$$y = X\beta + u$$

where

$$\begin{aligned}
u &= \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \text{ and } u_i \equiv \begin{pmatrix} v_i + w_{i1} \\ \vdots \\ v_i + w_{iT} \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} v_i + \begin{pmatrix} w_{i1} \\ \vdots \\ w_{iT} \end{pmatrix} \equiv l v_i + w_i\\
E[u] &= 0\\
Var(u) &= \begin{pmatrix} \Sigma & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Sigma \end{pmatrix} \text{ and } \Sigma = Var(u_i) = Var(lv_i + w_i) = \sigma_v^2 ll' + \sigma_w^2 I_T
\end{aligned}$$

Defining $\Omega = Var(u) = I_N \otimes \Sigma$, infeasible GLS estimator of β is

$$\widehat{\beta} = (X'\Omega X)^{-1} X'\Omega y$$

Show that we can write

$$\widehat{\beta} = (W_x + \theta B_x)^{-1}(W_x b_W + \theta B_x b_B)$$

where

$$\begin{aligned}
&\theta \text{ is some scalar.}\\
W_x &= \sum_i \sum_t (x_{it} - \overline{x}_i)(x_{it} - \overline{x}_i)'\\
W_{xy} &= \sum_i \sum_t (x_{it} - \overline{x}_i)(y_{it} - \overline{y}_i)\\
B_x &= \sum_i \sum_t \overline{x}_i \overline{x}_i' = T \sum_i \overline{x}_i \overline{x}_i'\\
B_{xy} &= \sum_i \sum_t \overline{x}_i \overline{y}_i = T \sum_i \overline{x}_i \overline{y}_i\\
b_W &= W_x^{-1} W_{xy} \text{ and } b_B = B_x^{-1} B_{xy}
\end{aligned}$$

Solution Note that

$$X'\Omega^{-1}X = \left(x_1' \cdots x_n'\right) \begin{pmatrix} \Sigma & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Sigma \end{pmatrix}^{-1} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_i x_i' \Sigma^{-1} x_i$$

Since[1]

$$\Sigma^{-1} = (\sigma_v^2 ll' + \sigma_w^2 I_T)^{-1} = \frac{1}{\sigma_w^2}\left(-\frac{\sigma_v^2}{T\sigma_v^2+\sigma_w^2} ll' + I_T\right)$$

we have

$$\begin{aligned}
X'\Omega^{-1}X &= \frac{1}{\sigma_w^2}\sum_i x_i'\left(-\frac{\sigma_v^2}{T\sigma_v^2+\sigma_w^2} ll' + I_T\right)x_i \\
&= \frac{1}{\sigma_w^2}\sum_i x_i'\left(I_T - \frac{\sigma_v^2+\frac{1}{T}\sigma_w^2}{T\sigma_v^2+\sigma_w^2} ll' + \frac{\frac{1}{T}\sigma_w^2}{T\sigma_v^2+\sigma_w^2} ll'\right)x_i \\
&= \frac{1}{\sigma_w^2}\sum_i x_i'\left(I_T - l\cdot\frac{1}{T}l' + \frac{\sigma_w^2}{T\sigma_v^2+\sigma_w^2} l\cdot\frac{1}{T}l'\right)x_i \\
&= \frac{1}{\sigma_w^2}\sum_i \left(x_i'\left[I_T - l(l'l)^{-1}l'\right]x_i + \frac{\sigma_w^2}{T\sigma_v^2+\sigma_w^2} x_i' l(l'l)^{-1}l'x_i\right) \\
&= \frac{1}{\sigma_w^2}\sum_i \left[\left(x_i - l\overline{x}_i'\right)'\left(x_i - l\overline{x}_i'\right) + \frac{\sigma_w^2}{T\sigma_v^2+\sigma_w^2}\left(l\overline{x}_i'\right)'\left(l\overline{x}_i'\right)\right] \\
&= \frac{1}{\sigma_w^2}\sum_i \left[\sum_t (x_{it}-\overline{x}_i)(x_i-\overline{x}_i)' + \frac{\sigma_w^2}{T\sigma_v^2+\sigma_w^2}\cdot T\overline{x}_i\overline{x}_i'\right] \\
&= \frac{1}{\sigma_w^2}\left(W_x + \frac{\sigma_w^2}{T\sigma_v^2+\sigma_w^2} B_x\right)
\end{aligned}$$

Also

$$\begin{aligned}
X'\Omega^{-1}y &= \frac{1}{\sigma_w^2}\sum_i \left[\sum_t (x_{it}-\overline{x}_i)(y_i-\overline{y}_i) + \frac{\sigma_w^2}{T\sigma_v^2+\sigma_w^2}\cdot T\overline{x}_i\overline{y}_i\right] \\
&= \frac{1}{\sigma_w^2}\left(W_{xy} + \frac{\sigma_w^2}{T\sigma_v^2+\sigma_w^2} B_{xy}\right) \\
&= \frac{1}{\sigma_w^2}\left(W_x b_W + \frac{\sigma_w^2}{T\sigma_v^2+\sigma_w^2} B_x b_B\right)
\end{aligned}$$

Define

$$\theta = \frac{\sigma_w^2}{T\sigma_v^2+\sigma_w^2}$$

then

$$\widehat{\beta} = (X'\Omega X)^{-1}X'\Omega y = (W_x + \theta B_x)^{-1}(W_x b_W + \theta B_x b_B)$$

[1] Let $\alpha ll' + \beta I_T$ be the inverse matrix of $all' + bI_T$. Since $I_T = (\alpha ll' + \beta I_T)(all' + bI) = (T\alpha a + \alpha b + \beta a)ll' + \beta b I_T$,

$$\beta = \frac{1}{b} \text{ and } \alpha = \frac{1}{b}\cdot\frac{-a}{Ta+b}$$