ECONOMETRICS
MONTE CARLO EXERCISE 1

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INCIDENTAL PARAMETER PROBLEM

- Model

$$y_{it} = \gamma y_{i,t-1} + \alpha_i + u_{it}$$

where

$$i = 1, \cdots, 500$$
$$t = 1, 2$$
$$u_{it} \sim iid\ N(0,1)$$
$$\alpha_i \sim N(0,1)$$
$$y_{i,0} \sim N\left(\frac{\alpha_i}{1-\gamma}, \frac{1}{1-\gamma^2}\right)$$

- Do OLS and IV

OLS uses 1st differential equation

$$y_{i,2} - y_{i,1} = \gamma(y_{i,1} - y_{i,0}) + (u_{i,2} - u_{i,1})$$

IV uses, for this model, the moment condition

$$E[y_{i,0}(u_{i,2} - u_{i,1})] = 0$$

ESTIMATION

- OLS uses the moment condition

$$E[(y_{i,1} - y_{i,0})(u_{i,2} - u_{i,1})] = 0$$

and thus

$$E\Big[(y_{i,1} - y_{i,0})\big\{(y_{i,2} - y_{i,1}) - \gamma(y_{i,1} - y_{i,0})\big\}\Big] = 0$$

So OLS estimator would be

$$\widehat{\gamma}_{OLS} = \frac{\sum_{i=1}^{500}(y_{i,1} - y_{i,0})(y_{i,2} - y_{i,1})}{\sum_{i=1}^{500}(y_{i,1} - y_{i,0})^2}$$

- IV has

$$E\Big[y_{i,0}\big\{(y_{i,2} - y_{i,1}) - \gamma(y_{i,1} - y_{i,0})\big\}\Big] = 0$$

so

$$\widehat{\gamma}_{IV} = \frac{\sum_{i=1}^{500} y_{i,0}(y_{i,2} - y_{i,1})}{\sum_{i=1}^{500} y_{i,0}(y_{i,1} - y_{i,0})}$$

STRENGTH OF AN INSTRUMENT

- The first stage regression looks like

$$y_{i,1} - y_{i,0} = \pi y_{i,0} + v_i$$

  and π indicates strength of the instrument.

- Theoretical π and R^2

$$\pi = \frac{E[(y_{i,1} - y_{i,0})y_{i,0}]}{E[y_{i,0}^2]} = -\frac{1}{2}(1-\gamma)^2$$

$$R^2 = \frac{Var(\pi y_{i,0})}{Var(y_{i,1} - y_{i,0})} = \frac{1}{4}(1-\gamma)^2$$

  As γ gets closer to 1, π goes to 0. (weak instrument)
  Also, R^2 becomes smaller (first stage doesn't explain variation)

PROOF

$$\pi = \frac{E[(y_{i,1} - y_{i,0})y_{i,0}]}{E[y_{i,0}^2]}$$
$$= \frac{E\Big[\{(\gamma - 1)y_{i,0} + \alpha_i + u_{i,1}\}y_{i,0}\Big]}{E[y_{i,0}^2]}$$
$$= \gamma - 1 + \frac{E[\alpha y_{i,0}]}{E[y_{i,0}^2]}$$
$$= \gamma - 1 + \frac{1 - \gamma^2}{2} = -\frac{1}{2}(1 - \gamma)^2$$

$$R^2 = \frac{\pi^2 Var(y_{i,0})}{Var(y_{i,1} - y_{i,0})}$$
$$= \frac{\pi^2 Var(y_{i,0})}{(\gamma - 1)^2 Var(y_{i,0})}$$
$$= \frac{\pi^2}{(\gamma - 1)^2} = \frac{1}{4}(1 - \gamma)^2$$

PROOF (Supplements)

$$E[\alpha_i y_{i,0}] = E\Big[\alpha_i E[y_i, 0|\alpha_i]\Big] = E\left[\frac{\alpha_i^2}{1-\gamma}\right] = \frac{1}{1-\gamma}$$

$$\begin{aligned}
E[y_{i,0}^2] = Var(y_{i,0}) &= E\Big[Var(y_{i,0}|\alpha_i)\Big] + Var\Big(E[y_{i,0}|\alpha_i]\Big) \\
&= E\left[\frac{1}{1-\gamma^2}\right] + Var\left(\frac{\alpha_i}{1-\gamma}\right) \\
&= \frac{1}{1-\gamma^2} + \frac{E[\alpha_i^2]}{(1-\gamma)^2} \\
&= \frac{1}{1-\gamma^2} + \frac{1}{(1-\gamma)^2} = \frac{2}{(1-\gamma^2)(1-\gamma)}
\end{aligned}$$

$$\begin{aligned}
Var(y_{i,1} - y_{i,0}) &= Var\Big((\gamma-1)y_{i,0} + \alpha_i + u_{i,1}\Big) \\
&= (\gamma-1)^2 Var(y_{i,0}) + Var(\alpha_i) + Var(u_{i,1}) \\
&\qquad + 2(\gamma-1)Cov(y_{i,0}, \alpha_i) \\
&= (\gamma-1)^2 Var(y_{i,0}) + 1 + 1 + 2(\gamma-1)E[\alpha_i y_{i,0}] \\
&= (\gamma-1)^2 Var(y_{i,0})
\end{aligned}$$

MONTE CARLO RESULTS

γ	Bias		RMSE		π	R^2
	OLS	IV	OLS	IV		
0	−0.5002	0.0040	0.5017	0.0900	−0.5	0.25
0.1	−0.5506	0.0074	0.5520	0.1059	−0.405	0.2025
0.2	−0.6004	0.0068	0.6018	0.1275	−0.32	0.16
0.3	−0.6494	0.0139	0.6508	0.1501	−0.245	0.1225
0.4	−0.7007	0.0161	0.7020	0.1860	−0.18	0.09
0.5	−0.7510	0.0281	0.7522	0.2395	−0.125	0.0625
0.6	−0.8002	0.0506	0.8014	0.3625	−0.08	0.04
0.7	−0.8502	0.1063	0.8514	0.6910	−0.045	0.0225
0.8	−0.8989	0.5558	0.8999	19.8141	−0.02	0.01
0.9	−0.9495	1.1511	0.9506	79.0939	−0.005	0.0025

Simulation number = 5,000

Bias$= \frac{1}{5,000}\sum_{s=1}^{5,000}(\widehat{\gamma}_s - \gamma)$ and RMSE$= \sqrt{\frac{1}{5,000}\sum_{s=1}^{5,000}(\widehat{\gamma}_s - \gamma)^2}$