ECONOMETRICS
MONTE CARLO EXERCISE 1

presented by
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INCIDENTAL PARAMETER PROBLEM

• Model

\[ y_{it} = \gamma y_{i,t-1} + \alpha_i + u_{it} \]

where

\[ i = 1, \cdots, 500 \]
\[ t = 1, 2 \]
\[ u_{it} \sim iid \ N(0, 1) \]
\[ \alpha_i \sim N(0, 1) \]
\[ y_{i,0} \sim N \left( \frac{\alpha_i}{1 - \gamma}, \frac{1}{1 - \gamma^2} \right) \]

• Do OLS and IV

OLS uses 1st differential equation

\[ y_{i,2} - y_{i,1} = \gamma(y_{i,1} - y_{i,0}) + (u_{i,2} - u_{i,1}) \]

IV uses, for this model, the moment condition

\[ E[y_{i,0}(u_{i,2} - u_{i,1})] = 0 \]
ESTIMATION

• OLS uses the moment condition

\[ E[(y_{i,1} - y_{i,0})(u_{i,2} - u_{i,1})] = 0 \]

and thus

\[ E\left[ (y_{i,1} - y_{i,0}) \{ (y_{i,2} - y_{i,1}) - \gamma(y_{i,1} - y_{i,0}) \} \right] = 0 \]

So OLS estimator would be

\[ \hat{\gamma}_{OLS} = \frac{\sum_{i=1}^{500} (y_{i,1} - y_{i,0})(y_{i,2} - y_{i,1})}{\sum_{i=1}^{500} (y_{i,1} - y_{i,0})^2} \]

• IV has

\[ E\left[ y_{i,0} \{ (y_{i,2} - y_{i,1}) - \gamma(y_{i,1} - y_{i,0}) \} \right] = 0 \]

so

\[ \hat{\gamma}_{IV} = \frac{\sum_{i=1}^{500} y_{i,0}(y_{i,2} - y_{i,1})}{\sum_{i=1}^{500} y_{i,0}(y_{i,1} - y_{i,0})} \]
STRENGTH OF AN INSTRUMENT

• The first stage regression looks like

\[ y_{i,1} - y_{i,0} = \pi y_{i,0} + u_i \]

and \( \pi \) indicates strength of the instrument.

• Theoretical \( \pi \) and \( R^2 \)

\[
\pi = \frac{E[(y_{i,1} - y_{i,0})y_{i,0}]}{E[y_{i,0}^2]} = -\frac{1}{2}(1 - \gamma)^2
\]

\[
R^2 = \frac{Var(\pi y_{i,0})}{Var(y_{i,1} - y_{i,0})} = \frac{1}{4}(1 - \gamma)^2
\]

As \( \gamma \) gets closer to 1, \( \pi \) goes to 0. (weak instrument)

Also, \( R^2 \) becomes smaller (first stage doesn’t explain variation)
PROOF

\[ \pi = \frac{E[(y_{i,1} - y_{i,0})y_{i,0}]}{E[y_{i,0}^2]} \]

\[ = \frac{E[\{(\gamma - 1)y_{i,0} + \alpha_i + u_{i,1}\}y_{i,0}]}{E[y_{i,0}^2]} \]

\[ = \gamma - 1 + \frac{E[\alpha y_{i,0}]}{E[y_{i,0}^2]} \]

\[ = \gamma - 1 + \frac{1 - \gamma^2}{2} = -\frac{1}{2}(1 - \gamma)^2 \]

\[ R^2 = \frac{\pi^2 Var(y_{i,0})}{Var(y_{i,1} - y_{i,0})} \]

\[ = \frac{\pi^2 Var(y_{i,0})}{(\gamma - 1)^2 Var(y_{i,0})} \]

\[ = \frac{\pi^2}{(\gamma - 1)^2} = \frac{1}{4}(1 - \gamma)^2 \]
PROOF (Supplements)

\[E[\alpha_iy_{i,0}] = E[\alpha_iE[y_{i,0}|\alpha_i]] = E\left[\frac{\alpha_i^2}{1-\gamma}\right] = \frac{1}{1-\gamma}\]

\[E[y_{i,0}^2] = \text{Var}(y_{i,0}) = E[\text{Var}(y_{i,0}|\alpha_i)] + \text{Var}\left(\frac{\alpha_i}{1-\gamma}\right) = \frac{1}{1-\gamma^2} + \frac{E[\alpha_i^2]}{(1-\gamma)^2} = \frac{1}{1-\gamma^2} + \frac{1}{(1-\gamma)^2} = \frac{2}{(1-\gamma^2)(1-\gamma)}\]

\[\text{Var}(y_{i,1} - y_{i,0}) = \text{Var}\left((\gamma - 1)y_{i,0} + \alpha_i + u_{i,1}\right) = (\gamma - 1)^2\text{Var}(y_{i,0}) + \text{Var}(\alpha_i) + \text{Var}(u_{i,1}) + 2(\gamma - 1)\text{Cov}(y_{i,0}, \alpha_i) = (\gamma - 1)^2\text{Var}(y_{i,0}) + 1 + 1 + 2(\gamma - 1)E[\alpha_i y_{i,0}] = (\gamma - 1)^2\text{Var}(y_{i,0})\]
### MONTE CARLO RESULTS

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>Bias OLS</th>
<th>Bias IV</th>
<th>RMSE OLS</th>
<th>RMSE IV</th>
<th>$\pi$</th>
<th>$R^2$</th>
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<tr>
<td>0</td>
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<td>-0.005</td>
<td>0.0025</td>
</tr>
</tbody>
</table>

Simulation number = 5,000

Bias = \( \frac{1}{5,000} \sum_{s=1}^{5,000} (\hat{\gamma}_s - \gamma) \) and RMSE = \( \sqrt{\frac{1}{5,000} \sum_{s=1}^{5,000} (\hat{\gamma}_s - \gamma)^2} \)