Econometrics GMM Exercise

Consider the model

$$y_i = x_i\beta + \varepsilon_i$$

where

$$E[x_i\varepsilon_i] = E[z_i\varepsilon_i] = 0$$

Show that the asymptotic variance of the GMM estimator for the moment equation

$$E\begin{bmatrix} x_i(y_i - x_i\beta) \\ z_i(y_i - x_i\beta) \end{bmatrix} = 0$$

is equal to

$$\sigma^2\left(E\begin{bmatrix} x_i^2 & x_iz_i \end{bmatrix}\left(E\begin{bmatrix} x_i^2 & x_iz_i \\ x_iz_i & z_i^2 \end{bmatrix}\right)^{-1}E\begin{bmatrix} x_i^2 \\ x_iz_i \end{bmatrix}\right)^{-1}$$

if it happens that

$$E\begin{bmatrix} x_i^2\varepsilon_i^2 & x_iz_i\varepsilon_i^2 \\ x_iz_i\varepsilon_i^2 & z_i^2\varepsilon_i^2 \end{bmatrix} = \sigma^2 E\begin{bmatrix} x_i^2 & x_iz_i \\ x_iz_i & z_i^2 \end{bmatrix}$$

Show that

$$E\begin{bmatrix} x_i^2 & x_iz_i \end{bmatrix}\left(E\begin{bmatrix} x_i^2 & x_iz_i \\ x_iz_i & z_i^2 \end{bmatrix}\right)^{-1}E\begin{bmatrix} x_i^2 \\ x_iz_i \end{bmatrix} = E[x_i^2]$$

Conclude that the asymptotic variance is equal t

$$\frac{\sigma^2}{E[x_i^2]}$$

Convince yourself that it is the asymptotic variance ofthe OLS estimator.

Solution GMM estimator solves the following problem.

$$\min_{\beta}\left(\frac{1}{n}\sum_{i=1}^{n}\begin{bmatrix} x_i(y_i - x_i\beta) \\ z_i(y_i - x_i\beta) \end{bmatrix}\right)' A\left(\frac{1}{n}\sum_{i=1}^{n}\begin{bmatrix} x_i(y_i - x_i\beta) \\ z_i(y_i - x_i\beta) \end{bmatrix}\right)$$

Optimal GMM uses the weight matrix

$$\begin{aligned} A &= \left[nVar\left(\frac{1}{n}\sum_{i=1}^{n}\begin{bmatrix} x_i(y_i - x_i\beta) \\ z_i(y_i - x_i\beta) \end{bmatrix}\right)\right]^{-1} \\ &= \left[Var\begin{pmatrix} x_i\varepsilon_i \\ z_i\varepsilon_i \end{pmatrix}\right]^{-1} \\ &= \left(E\left[\begin{pmatrix} x_i\varepsilon_i \\ z_i\varepsilon_i \end{pmatrix}\begin{pmatrix} x_i\varepsilon_i & z_i\varepsilon_i \end{pmatrix}\right]\right)^{-1} \\ &= \left(E\begin{bmatrix} x_i^2\varepsilon_i^2 & x_iz_i\varepsilon_i^2 \\ x_iz_i\varepsilon_i^2 & z_i^2\varepsilon_i^2 \end{bmatrix}\right)^{-1} = \left(\sigma^2 E\begin{bmatrix} x_i^2 & x_iz_i \\ x_iz_i & z_i^2 \end{bmatrix}\right)^{-1} \end{aligned}$$

The first order condition of the minization problem is

$$2\left(\frac{1}{n}\sum_{i=1}^{n}\left[\begin{array}{c} -x_i^2 \\ -z_i x_i \end{array}\right]\right)' A \left(\frac{1}{n}\sum_{i=1}^{n}\left[\begin{array}{c} x_i(y_i - x_i\beta) \\ z_i(y_i - x_i\beta) \end{array}\right]\right) = 0$$

Arranging the terms, the GMM estimator of β is

$$\widehat{\beta} = \left(\left[\frac{1}{n}\sum_{i=1}^{n}\left(\begin{array}{c} x_i^2 \\ z_i x_i \end{array}\right)\right]' A \left[\frac{1}{n}\sum_{i=1}^{n}\left(\begin{array}{c} x_i^2 \\ z_i x_i \end{array}\right)\right]\right)^{-1} \left[\frac{1}{n}\sum_{i=1}^{n}\left(\begin{array}{c} x_i^2 \\ z_i x_i \end{array}\right)\right]' A \left[\frac{1}{n}\sum_{i=1}^{n}\left(\begin{array}{c} x_i y_i \\ z_i y_i \end{array}\right)\right]$$

Using $y_i = x_i\beta + \varepsilon_i$,

$$\sqrt{n}(\widehat{\beta} - \beta) = \left(\left[\frac{1}{n}\sum_{i=1}^{n}\left(\begin{array}{c} x_i^2 \\ z_i x_i \end{array}\right)\right]' A \left[\frac{1}{n}\sum_{i=1}^{n}\left(\begin{array}{c} x_i^2 \\ z_i x_i \end{array}\right)\right]\right)^{-1} \left[\frac{1}{n}\sum_{i=1}^{n}\left(\begin{array}{c} x_i^2 \\ z_i x_i \end{array}\right)\right]' A \left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\begin{array}{c} x_i \varepsilon_i \\ z_i \varepsilon_i \end{array}\right)\right]$$

and

$$\sqrt{n}(\widehat{\beta} - \beta) \xrightarrow{d} N(0, V)$$

where

$$\begin{aligned} V &= (B'AB)^{-1}B'AVar\left(\begin{array}{c} x_i\varepsilon_i \\ z_i\varepsilon_i \end{array}\right)AB(B'AB)^{-1} \\ &= (B'AB)^{-1}B'AA^{-1}AB(B'AB)^{-1} \\ &= (B'AB)^{-1} \\ B &= E\left[\begin{array}{c} x_i^2 \\ z_i x_i \end{array}\right] \end{aligned}$$

So the asymptotic variance is

$$\begin{aligned} V &= \left(E\left[\begin{array}{cc} x_i^2 & x_i z_i \end{array}\right]\left(\sigma^2 E\left[\begin{array}{cc} x_i^2 & x_i z_i \\ x_i z_i & z_i^2 \end{array}\right]\right)^{-1} E\left[\begin{array}{c} x_i^2 \\ x_i z_i \end{array}\right]\right)^{-1} \\ &= \sigma^2\left(E\left[\begin{array}{cc} x_i^2 & x_i z_i \end{array}\right]\left(E\left[\begin{array}{cc} x_i^2 & x_i z_i \\ x_i z_i & z_i^2 \end{array}\right]\right)^{-1} E\left[\begin{array}{c} x_i^2 \\ x_i z_i \end{array}\right]\right)^{-1} \end{aligned}$$

Now

$$\left(E\left[\begin{array}{cc} x_i^2 & x_i z_i \\ x_i z_i & z_i^2 \end{array}\right]\right)^{-1} = \frac{1}{E[x_i^2]E[z_i^2] - (E[x_i z_i])^2}\left(\begin{array}{cc} E[z_i^2] & -E[x_i z_i] \\ -E[x_i z_i] & E[x_i^2] \end{array}\right)$$

so

$$\begin{aligned} V &= \sigma^2\left[\frac{1}{E[x_i^2]E[z_i^2] - (E[x_i z_i])^2}\left(\begin{array}{cc} E[x_i^2] & E[x_i z_i] \end{array}\right)\left(\begin{array}{c} E[x_i^2]E[z_i^2] - (E[x_i z_i])^2 \\ 0 \end{array}\right)\right]^{-1} \\ &= \frac{\sigma^2}{E[x_i^2]} \end{aligned}$$

This is the same with the asymptotic variance of the OLS estimator. Since the OLS estimator is $\widehat{\beta}_{OLS} = \left(\frac{1}{n}\sum_i x_i^2\right)^{-1}\left(\frac{1}{n}\sum_i x_i y_i\right)$, its asymptotic variance is

$$\left(E[x_i^2]\right)^{-1} Var(x_i\varepsilon_i)\left(E[x_i^2]\right)^{-1} = \left(E[x_i^2]\right)^{-1}\sigma^2 E[x_i^2]\left(E[x_i^2]\right)^{-1} = \frac{\sigma^2}{E[x_i^2]}$$