Econometrics Theoretical Exercise 2: by Yang, Yong Hyeon

Consider a simple model

$$y_{it} \overset{iid}{\sim} N(\alpha_{i0}, \theta_0), \qquad t = 1, \cdots, T, i = 1, \cdots, n$$

or

$$\log f(y_{it}; \theta, \alpha_i) = C - \frac{1}{2}\log\theta - \frac{(y_{it} - \alpha_i)^2}{2\theta}$$

Adopt simpler notations $\alpha_i(\theta, \epsilon) = \alpha_i(\theta, F_i(\epsilon))$ and $\theta(\epsilon) = \theta(F(\epsilon))$.

Q1. Obtain

$$\begin{aligned}
u_{it}(\theta, \alpha) &\equiv \frac{\partial}{\partial\theta}\log f(y_{it}; \theta, \alpha) \\
v_{it}(\theta, \alpha) &\equiv \frac{\partial}{\partial\alpha_i}\log f(y_{it}; \theta, \alpha) \\
u_{it}^{\alpha_i} &\equiv \frac{\partial}{\partial\alpha_i} u_{it}(\theta_0, \alpha_{i0}) \\
v_{it}^{\alpha_i} &\equiv \frac{\partial}{\partial\alpha_i} v_{it}(\theta_0, \alpha_{i0})
\end{aligned}$$

Conclude that

$$E[v_{it}^{\alpha_i}]^{-1} E[u_{it}^{\alpha_i}] = 0$$

and

$$U_{it}(\theta, \alpha) = -\frac{1}{2\theta} + \frac{(y_{it} - \alpha)^2}{2\theta^2}$$

Solution Differentiating the log likelihood,

$$\begin{aligned}
u_{it}(\theta, \alpha) &= \frac{\partial}{\partial\theta}\log f(y_{it}; \theta, \alpha) = -\frac{1}{2\theta} + \frac{(y_{it} - \alpha)^2}{2\theta^2} \\
v_{it}(\theta, \alpha) &= \frac{\partial}{\partial\alpha_i}\log f(y_{it}; \theta, \alpha) = \frac{y_{it} - \alpha}{\theta} \\
\frac{\partial}{\partial\alpha_i} u_{it}(\theta, \alpha) &= -\frac{y_{it} - \alpha}{\theta^2} \\
\frac{\partial}{\partial\alpha_i} v_{it}(\theta, \alpha) &= -\frac{1}{\theta}
\end{aligned}$$

So

$$\begin{aligned}
u_{it}^{\alpha_i} &= -\frac{y_{it} - \alpha_{i0}}{\theta_0^2} \\
v_{it}^{\alpha_i} &= -\frac{1}{\theta_0}
\end{aligned}$$

and thus

$$\begin{aligned}
E[u_{it}^{\alpha_i}] &= -\frac{E[y_{it} - \alpha_{i0}]}{\theta_0^2} = 0 \\
E[v_{it}^{\alpha_i}] &= -\frac{1}{\theta_0}
\end{aligned}$$

Therefore,

$$E[v_{it}^{\alpha_i}]^{-1}E[u_{it}^{\alpha_i}] = 0$$

and also

$$U_{it}(\theta,\alpha) = u_{it}(\theta,\alpha) - E[v_{it}^{\alpha_i}]^{-1}E[u_{it}^{\alpha_i}]v_{it}(\theta,\alpha) = u_{it}(\theta,\alpha) = -\frac{1}{2\theta} + \frac{(y_{it}-\alpha)^2}{2\theta^2}$$

Q2. Note that $\alpha_i(\theta,\epsilon)$ is a solution to

$$0 = \int \frac{y-\alpha}{\theta} dF_i(\epsilon) = \int \frac{y-\alpha}{\theta} F_i(dy) + \epsilon\sqrt{T}\int \frac{y-\alpha}{\theta}\left(\widehat{F}_i - F_i\right)(dy)$$

or multiplying both sides by θ, it is a solution to

$$\begin{aligned} 0 &= \left(1-\epsilon\sqrt{T}\right)\int (y-\alpha)F_i(dy) + \epsilon\sqrt{T}\int (y-\alpha)\widehat{F}_i(dy) \\ &= \left(1-\epsilon\sqrt{T}\right)\int (y-\alpha)\frac{1}{\sqrt{2\pi\theta_0}}\exp\left(-\frac{[y-\alpha_{i0}]^2}{2\theta_0}\right)dy + \epsilon\sqrt{T}\frac{1}{T}\sum_{t=1}^{T}(y_{it}-\alpha) \end{aligned}$$

Show that this can be equivalently written as

$$0 = -(\alpha - \alpha_{i0}) + \epsilon\sqrt{T}\overline{u}_i$$

where

$$u_{it} \equiv y_{it} - \alpha_{i0}$$

Conclude that

$$\alpha_i(\theta,\epsilon) = \alpha_{i0} + \epsilon\sqrt{T}\overline{u}_i$$

Solution From the second equation, $\alpha_i(\theta,\epsilon)$ is a solution to

$$\begin{aligned} 0 &= \left(1-\epsilon\sqrt{T}\right)E[y-\alpha] + \epsilon\sqrt{T}\frac{1}{T}\sum_{t=1}^{T}(y_{it}-\alpha) \\ &= \left(1-\epsilon\sqrt{T}\right)E[y-\alpha_{i0}+\alpha_{i0}-\alpha] + \epsilon\sqrt{T}\frac{1}{T}\sum_{t=1}^{T}(y_{it}-\alpha_{i0}+\alpha_{i0}-\alpha) \\ &= \left(1-\epsilon\sqrt{T}\right)E[y-\alpha_{i0}] + \left(1-\epsilon\sqrt{T}\right)(\alpha_{i0}-\alpha) + \epsilon\sqrt{T}\frac{1}{T}\sum_{t=1}^{T}(y-\alpha_{i0}) + \epsilon\sqrt{T}(\alpha_{i0}-\alpha) \\ &= (\alpha_{i0}-\alpha) + \epsilon\sqrt{T}\frac{1}{T}\sum_{t=1}^{T}u_{it} \\ &= -(\alpha-\alpha_{i0}) + \epsilon\sqrt{T}\overline{u}_i \end{aligned}$$

Hence,

$$\alpha_i(\theta,\epsilon) = \alpha_{i0} + \epsilon\sqrt{T}\overline{u}_i$$

Q3. Note that $\theta(\epsilon) = \theta(F(\epsilon))$ is the solution to

$$
\begin{aligned}
0 &= \sum_{i=1}^{n} \int \left(-\frac{1}{2\theta} + \frac{[y_{it} - \alpha_i(\theta,\epsilon)]^2}{2\theta^2} \right) dF_i(\epsilon) \\
&= \left(1 - \epsilon\sqrt{T}\right) \sum_{i=1}^{n} \int \left(-\frac{1}{2\theta} + \frac{[y - \alpha_i(\theta,\epsilon)]^2}{2\theta^2} \right) F_i(dy) + \epsilon\sqrt{T} \sum_{i=1}^{n} \int \left(-\frac{1}{2\theta} + \frac{[y - \alpha_i(\theta,\epsilon)]^2}{2\theta^2} \right) \widehat{F}_i(dy)
\end{aligned}
$$

Multiplying both sides by $2\theta^2$, we can conclude that it is the solution to

$$
0 = \left(1 - \epsilon\sqrt{T}\right) \sum_{i=1}^{n} \int \left(-\theta + [y - \alpha_i(\theta,\epsilon)]^2\right) F_i(dy) + \epsilon\sqrt{T} \sum_{i=1}^{n} \int \left(-\theta + [y - \alpha_i(\theta,\epsilon)]^2\right) \widehat{F}_i(dy)
$$

(a) Show that

$$
\int \left[-\theta + (y - \alpha)^2\right] F_i(dy) = -(\theta - \theta_0) + (\alpha - \alpha_{i0})^2
$$

Solution

$$
\begin{aligned}
\int \left[-\theta + (y - \alpha)^2\right] F_i(dy) &= E\left[-\theta + (y - \alpha)^2\right] \\
&= E\left[-\theta + (y - \alpha_{i0} + \alpha_{i0} - \alpha)^2\right] \\
&= E\left[-\theta + (y - \alpha_{i0})^2 + (\alpha_{i0} - \alpha)^2 + 2(y - \alpha_{i0})(\alpha_{i0} - \alpha)\right] \\
&= -\theta + \theta_0 + (\alpha_{i0} - \alpha)^2 + 2E[y - \alpha_{i0}](\alpha_{i0} - \alpha) \\
&= -(\theta - \theta_0) + (\alpha - \alpha_{i0})^2
\end{aligned}
$$

(b) Show that

$$
\int \left[-\theta + (y - \alpha)^2\right] \widehat{F}_i(dy) = -(\theta - \theta_0) + \frac{1}{T}\sum_{t=1}^{T}(u_{it}^2 - \theta_0) - 2(\alpha - \alpha_{i0})\overline{u}_i + (\alpha - \alpha_{i0})^2
$$

Solution

$$
\begin{aligned}
\int \left[-\theta + (y - \alpha)^2\right] \widehat{F}_i(dy) &= \frac{1}{T}\sum_{t=1}^{T}\left[-\theta + (y - \alpha)^2\right] \\
&= \frac{1}{T}\sum_{t=1}^{T}\left[-\theta + (y - \alpha_{i0} + \alpha_{i0} - \alpha)^2\right] \\
&= \frac{1}{T}\sum_{t=1}^{T}\left[-\theta + (y - \alpha_{i0})^2 + 2(y - \alpha_{i0})(\alpha_{i0} - \alpha) + (\alpha_{i0} - \alpha)^2\right] \\
&= -\theta + \frac{1}{T}\sum_{t=1}^{T} u_{it}^2 + \frac{1}{T}\sum_{t=1}^{T} 2u_{it}(\alpha_{i0} - \alpha) + (\alpha_{i0} - \alpha)^2 \\
&= -(\theta - \theta_0) + \frac{1}{T}\sum_{t=1}^{T}(u_{it}^2 - \theta_0) - 2(\alpha - \alpha_{i0})\overline{u}_i + (\alpha - \alpha_{i0})^2
\end{aligned}
$$

(c) Combining (a) and (b), conclude that

$$\left(1-\epsilon\sqrt{T}\right)\sum_{i=1}^{n}\int\left[-\theta+(y-\alpha)^2\right]F_i(dy)+\epsilon\sqrt{T}\sum_{i=1}^{n}\int\left[-\theta+(y-\alpha)^2\right]\widehat{F}_i(dy)$$
$$=-n(\theta-\theta_0)+\sum_{i=1}^{n}(\alpha-\alpha_{i0})^2+\frac{\epsilon}{\sqrt{T}}\sum_{i=1}^{n}\sum_{t=1}^{T}(u_{it}^2-\theta_0)-2\epsilon\sum_{i=1}^{n}(\alpha-\alpha_{i0})\left(\sqrt{T}\overline{u}_i\right)$$

Solution Since

$$\left(1-\epsilon\sqrt{T}\right)\sum_{i=1}^{n}\int\left[-\theta+(y-\alpha)^2\right]F_i(dy)=-n\left(1-\epsilon\sqrt{T}\right)(\theta-\theta_0)+\left(1-\epsilon\sqrt{T}\right)\sum_{i=1}^{n}(\alpha-\alpha_{i0})^2$$

and

$$\epsilon\sqrt{T}\sum_{i=1}^{n}\int\left[-\theta+(y-\alpha)^2\right]\widehat{F}_i(dy)$$
$$=-n\epsilon\sqrt{T}(\theta-\theta_0)+\epsilon\sqrt{T}\frac{1}{T}\sum_{i=1}^{n}\sum_{t=1}^{T}(u_{it}^2-\theta_0)-2\epsilon\sqrt{T}\sum_{i=1}^{n}(\alpha-\alpha_{i0})\overline{u}_i+\epsilon\sqrt{T}\sum_{i=1}^{n}(\alpha-\alpha_{i0})^2$$

we have

$$\left(1-\epsilon\sqrt{T}\right)\sum_{i=1}^{n}\int\left[-\theta+(y-\alpha)^2\right]F_i(dy)+\epsilon\sqrt{T}\sum_{i=1}^{n}\int\left[-\theta+(y-\alpha)^2\right]\widehat{F}_i(dy)$$
$$=-n(\theta-\theta_0)+\epsilon\sqrt{T}\frac{1}{T}\sum_{i=1}^{n}\sum_{t=1}^{T}(u_{it}^2-\theta_0)-2\epsilon\sqrt{T}\sum_{i=1}^{n}(\alpha-\alpha_{i0})\overline{u}_i+\sum_{i=1}^{n}(\alpha-\alpha_{i0})^2$$
$$=-n(\theta-\theta_0)+\sum_{i=1}^{n}(\alpha-\alpha_{i0})^2+\frac{\epsilon}{\sqrt{T}}\sum_{i=1}^{n}\sum_{t=1}^{T}(u_{it}^2-\theta_0)-2\epsilon\sum_{i=1}^{n}(\alpha-\alpha_{i0})\left(\sqrt{T}\overline{u}_i\right)$$

(d) Plug $\alpha_i(\theta,\epsilon)-\alpha_{i0}=\epsilon\sqrt{T}\overline{u}_i$, and conclude that

$$\left(1-\epsilon\sqrt{T}\right)\sum_{i=1}^{n}\int\left(-\theta+[y-\alpha_i(\theta,\epsilon)]^2\right)F_i(dy)+\epsilon\sqrt{T}\sum_{i=1}^{n}\int\left(-\theta+[y-\alpha_i(\theta,\epsilon)]^2\right)\widehat{F}_i(dy)$$
$$=-n(\theta-\theta_0)+\frac{\epsilon}{\sqrt{T}}\sum_{i=1}^{n}\sum_{t=1}^{T}(u_{it}^2-\theta_0)-\epsilon^2\sum_{i=1}^{n}\left(\sqrt{T}\overline{u}_i\right)^2$$

Solution

$$\left(1-\epsilon\sqrt{T}\right)\sum_{i=1}^{n}\int\left(-\theta+[y-\alpha_i(\theta,\epsilon)]^2\right)F_i(dy)+\epsilon\sqrt{T}\sum_{i=1}^{n}\int\left(-\theta+[y-\alpha_i(\theta,\epsilon)]^2\right)\widehat{F}_i(dy)$$
$$=-n(\theta-\theta_0)+\sum_{i=1}^{n}\epsilon^2\left(\sqrt{T}\overline{u}_i\right)^2+\frac{\epsilon}{\sqrt{T}}\sum_{i=1}^{n}\sum_{t=1}^{T}(u_{it}^2-\theta_0)-2\epsilon\sum_{i=1}^{n}\epsilon\left(\sqrt{T}\overline{u}_i\right)^2$$
$$=-n(\theta-\theta_0)+\frac{\epsilon}{\sqrt{T}}\sum_{i=1}^{n}\sum_{t=1}^{T}(u_{it}^2-\theta_0)-\epsilon^2\sum_{i=1}^{n}\left(\sqrt{T}\overline{u}_i\right)^2$$

(e) Conclude that

$$\theta(\epsilon)-\theta_0=\frac{\epsilon}{n\sqrt{T}}\sum_{i=1}^{n}\sum_{t=1}^{T}(u_{it}^2-\theta_0)-\frac{\epsilon^2}{n}\sum_{i=1}^{n}\left(\sqrt{T}\overline{u}_i\right)^2$$

Solution Now that $\theta(\epsilon)$ is the solution to

$$0=-n(\theta-\theta_0)+\frac{\epsilon}{\sqrt{T}}\sum_{i=1}^{n}\sum_{t=1}^{T}(u_{it}^2-\theta_0)-\epsilon^2\sum_{i=1}^{n}\left(\sqrt{T}\overline{u}_i\right)^2$$

So

$$\theta(\epsilon)-\theta_0=\frac{\epsilon}{n\sqrt{T}}\sum_{i=1}^{n}\sum_{t=1}^{T}(u_{it}^2-\theta_0)-\frac{\epsilon^2}{n}\sum_{i=1}^{n}\left(\sqrt{T}\overline{u}_i\right)^2 \tag{1}$$

(f) Verify that

$$\theta(0)=\theta_0$$
$$\theta\left(\frac{1}{\sqrt{T}}\right)=\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}(y_{it}-\overline{y}_i)^2$$

Solution Plug $\epsilon=0$ in (1), then

$$\theta(0)-\theta_0=0$$

or equivalently,

$$\theta(0)=\theta_0$$

Also, plug $\epsilon=\frac{1}{\sqrt{T}}$ in (1), then

$$\theta\left(\frac{1}{\sqrt{T}}\right)-\theta_0=\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}(u_{it}^2-\theta_0)-\frac{1}{nT}\sum_{i=1}^{n}\left(\sqrt{T}\overline{u}_i\right)^2$$

and thus

$$\begin{aligned}\theta\left(\frac{1}{\sqrt{T}}\right)&=\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}u_{it}^2-\frac{1}{nT}\sum_{i=1}^{n}T\overline{u}_i^2\\&=\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}(u_{it}^2-\overline{u}_i^2)\\&=\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}(u_{it}-\overline{u}_i)^2\\&=\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}[(y_{it}-\alpha_{i0})-(\overline{y}_i-\alpha_{i0})]^2\\&=\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}(y_{it}-\overline{y}_i)^2\end{aligned}$$

(g) Show that

$$\theta^{\epsilon}(0) = \frac{1}{n\sqrt{T}} \sum_{i=1}^{n} \sum_{t=1}^{T} (u_{it}^2 - \theta_0)$$
$$\frac{\theta^{\epsilon\epsilon}(0)}{2} = -\frac{1}{n} \sum_{i=1}^{n} \left(\sqrt{T}\overline{u}_i\right)^2$$

Solution Take a derivative of (1) with respect to ϵ, then

$$\theta^{\epsilon}(\epsilon) = \frac{1}{n\sqrt{T}} \sum_{i=1}^{n} \sum_{t=1}^{T} (u_{it}^2 - \theta_0) - \frac{2\epsilon}{n} \sum_{i=1}^{n} \left(\sqrt{T}\overline{u}_i\right)^2$$

and thus

$$\theta^{\epsilon}(0) = \frac{1}{n\sqrt{T}} \sum_{i=1}^{n} \sum_{t=1}^{T} (u_{it}^2 - \theta_0)$$

Take a derivative once more, then

$$\theta^{\epsilon\epsilon}(\epsilon) = -\frac{2}{n} \sum_{i=1}^{n} \left(\sqrt{T}\overline{u}_i\right)^2$$

Therefore,

$$\frac{\theta^{\epsilon\epsilon}(0)}{2} = -\frac{1}{n} \sum_{i=1}^{n} \left(\sqrt{T}\overline{u}_i\right)^2$$

(h) Show that

$$\frac{\theta^{\epsilon\epsilon}(0)}{2} = -\theta_0 + o_p(1)$$

Solution Note that

$$\overline{u}_i = \frac{1}{T} \sum_{t=1}^{T} u_{it} \sim N\left(0, \frac{1}{T}\theta_0\right)$$

and thus

$$\left(\frac{\sqrt{T}\overline{u}_i}{\sqrt{\theta_0}}\right)^2 \sim \chi_1^2$$

Note also that $(\sqrt{T}\overline{u}_i)^2$ is iid over i in this model. Use weak LLN to obtain

$$\frac{1}{n} \sum_{i=1}^{n} \left(\sqrt{T}\overline{u}_i\right)^2 = E\left[\left(\sqrt{T}\overline{u}_i\right)^2\right] + o_p(1) = \theta_0 E\left[\left(\frac{\sqrt{T}\overline{u}_i}{\sqrt{\theta_0}}\right)^2\right] + o_p(1) = \theta_0 + o_p(1)$$

Therefore,

$$\frac{\theta^{\epsilon\epsilon}(0)}{2} = -\theta_0 + o_p(1)$$

Additional Comments We can solve (h) without deriving the exact distribution of $\overline{u}_i$. However, this method is confusing. Note first that since u_{it} is iid over t,

$$E\left[\left(\sqrt{T}\overline{u}_i\right)^2\right] = E\left[T\overline{u}_i^2\right] = E[u_{it}^2] = \theta_0$$

Also

$$\begin{aligned}
Var\left(\left[\sqrt{T}\overline{u}_i\right]^2\right) &= Var\left(T\overline{u}_i^2\right) \\
&= Var\left(\frac{1}{T}\sum_{t=1}^{T} u_{it}^2 + \frac{1}{T}\sum_{s\neq t} u_{is}u_{it}\right) \\
&= \frac{1}{T}Var(u_{it}^2) + \frac{1}{T^2}\sum_{p\neq q}\sum_{s\neq t} Cov(u_{ip}u_{iq}, u_{is}u_{it}) \qquad \because u_{it} \text{ is iid over } i, t
\end{aligned}$$

The first term is

$$\frac{1}{T}Var(u_{it}^2) = \frac{\theta_0^2}{T}Var\left(\left[\frac{u_{it}}{\sqrt{\theta_0}}\right]^2\right) = \frac{2\theta_0^2}{T}$$

since $\left[\frac{u_{it}}{\sqrt{\theta_0}}\right]^2 \sim \chi_1^2$. Now consider the second term.

$$\begin{aligned}
\sum_{p\neq q}\sum_{s\neq t} Cov(u_{ip}u_{iq}, u_{is}u_{it}) &= \sum_{p\neq q}\sum_{s\neq t}\Big(E[u_{ip}u_{iq}u_{is}u_{it}] - E[u_{ip}u_{iq}]E[u_{is}u_{it}]\Big) \\
&= \sum_{p\neq q}\sum_{s\neq t} E[u_{ip}u_{iq}u_{is}u_{it}] \\
&= \sum_{p=s\neq q=t} E[u_{ip}u_{iq}u_{is}u_{it}] + \sum_{p=t\neq q=s} E[u_{ip}u_{iq}u_{is}u_{it}] \\
&= 2\sum_{s\neq t} E[u_{is}^2 u_{it}^2] \\
&= 2\sum_{s\neq t} E[u_{is}^2]E[u_{it}^2] \\
&= 2T(T-1)\theta_0^2
\end{aligned}$$

Therefore,

$$Var\left(\left[\sqrt{T}\overline{u}_i\right]^2\right) = \frac{2\theta_0^2}{T} + \frac{2(T-1)\theta_0^2}{T} = 2\theta_0^2 < \infty$$

Since $(\sqrt{T}\overline{u}_i)^2$ is iid over i, apply weak LLN to write

$$\frac{\theta^{\epsilon\epsilon}(0)}{2} \xrightarrow{p} -E\left[\left(\sqrt{T}\overline{u}_i\right)^2\right] = -\theta_0$$

or equivalently,

$$\frac{\theta^{\epsilon\epsilon}(0)}{2} = -\theta_0 + o_p(1)$$