Econometrics Theoretical Exercise 3: by Yang, Yong Hyeon

Consider a simple model where

$$\begin{aligned} y_{it} &= \theta_0 x_{it} + \varepsilon_{it} \\ x_{it} &= \alpha_{i0} + w_{it}, \quad (t = 1, \cdots, T; i = 1, \cdots, n) \end{aligned}$$

For simplicity, we will assume that $(\varepsilon_{it}, w_{it})$ is normal, and iid over i and t. We consdier the 2SLS for θ, which can be characterized by the following:

$$\sum_{i=1}^{n} \sum_{t=1}^{T} u\left(z_{it}; \widehat{\theta}, \widehat{\alpha}_i\right) = 0$$

$$\sum_{t=1}^{T} v\left(z_{it}; \widehat{\alpha}_i\right) = 0$$

where $z_{it} = (y_{it}, x_{it})$ and

$$\begin{aligned} u(z_{it}; \theta, \alpha_i) &= \alpha_i(y_{it} - \theta\alpha_i) \equiv u_{it}(\theta, \alpha) \\ v(z_{it}; \alpha_i) &= x_{it} - \alpha_i \equiv v_{it}(\alpha) \end{aligned}$$

Q1. Show that

$$\begin{aligned} v_{it}^{\alpha_i} &= -1 \\ u_{it}^{\alpha_i} &= y_{it} - 2\theta_0\alpha_{i0} \end{aligned}$$

Conclude that

$$E[v_{it}^{\alpha_i}]^{-1} E[u_{it}^{\alpha_i}] = \theta_0\alpha_{i0}$$

and

$$U_{it}(\theta, \alpha_i) = \alpha_i(y_{it} - \theta\alpha_i) - \theta_0\alpha_{i0}(x_{it} - \alpha_i)$$

Verify that $U_{it}^{\alpha_i} \equiv U_{it}^{\alpha_i}(\theta_0, \alpha_{i0})$ has a zero expectation.

Solution Note first

$$v_{it}^{\alpha_i}(\alpha_i) = -1$$

So

$$v_{it}^{\alpha_i} \equiv v_{it}^{\alpha_i}(\alpha_{i0}) = -1$$

Also

$$u_{it}^{\alpha_i}(\theta, \alpha_i) = y_{it} - 2\theta\alpha_i$$

and thus

$$u_{it}^{\alpha_i} \equiv u_{it}^{\alpha_i}(\theta_0, \alpha_{i0}) = y_{it} - 2\theta_0\alpha_{i0}$$

Therefore

$$\frac{E[u_{it}^{\alpha_i}]}{E[v_{it}^{\alpha_i}]} = \frac{E[y_{it} - 2\theta_0\alpha_{i0}]}{E[-1]} = \frac{\theta_0\alpha_{i0} - 2\theta_0\alpha_{i0}}{-1} = \theta_0\alpha_{i0}$$

Define

$$U_{it}(\theta,\alpha_i) \equiv u_{it}(\theta,\alpha_i) - E[v_{it}^{\alpha_i}]^{-1}E[u_{it}^{\alpha_i}]v_{it}(\alpha_i) = \alpha_i(y_{it}-\theta\alpha_i) - \theta_0\alpha_{i0}(x_{it}-\alpha_i) \quad (1)$$

Then,

$$U_{it}^{\alpha_i}(\theta,\alpha_i) = y_{it} - 2\theta\alpha_i + \theta_0\alpha_{i0} \quad (2)$$

So

$$E[U_{it}^{\alpha_i}] = E[U_{it}^{\alpha_i}(\theta_0,\alpha_{i0})] = E[y_{it}] - 2\theta_0\alpha_{i0} + \theta_0\alpha_{i0} = 0$$

Q2. Prove that $U_{it}^{\theta} \equiv U_{it}^{\theta}(\theta_0,\alpha_{i0})$ has a characterization

$$U_{it}^{\theta} = -\alpha_{i0}^2$$

Conclude that

$$-E[U_{it}^{\theta}] = \alpha_{i0}^2$$

Solution From (1),

$$U_{it}^{\theta}(\theta,\alpha_i) = -\alpha_i^2$$

So

$$U_{it}^{\theta} \equiv U_{it}^{\theta}(\theta_0,\alpha_{i0}) = -\alpha_{i0}^2$$

and thus

$$-E[U_{it}^{\theta}] = \alpha_{i0}^2$$

Q3. Prove that

$$\begin{aligned} U_{it}^{\alpha_i}(\theta,\alpha_i) &= y_{it} - 2\theta\alpha_i + \theta_0\alpha_{i0} \\ U_{it}^{\alpha_i\alpha_i}(\theta,\alpha_i) &= -2\theta \end{aligned}$$

Conclude that

$$\begin{aligned} U_{it}^{\alpha_i} &= y_{it} - \theta_0\alpha_{i0} \\ U_{it}^{\alpha_i\alpha_i} &= -2\theta_0 \end{aligned}$$

Solution Recall (2).

$$\begin{aligned} U_{it}^{\alpha_i}(\theta,\alpha_i) &= y_{it} - 2\theta\alpha_i + \theta_0\alpha_{i0} \\ U_{it}^{\alpha_i\alpha_i}(\theta,\alpha_i) &= -2\theta \end{aligned}$$

So

$$\begin{aligned} U_{it}^{\alpha_i} &= y_{it} - 2\theta_0\alpha_{i0} + \theta_0\alpha_{i0} = y_{it} - \theta_0\alpha_{i0} \\ U_{it}^{\alpha_i\alpha_i} &\equiv U_{it}^{\alpha_i\alpha_i}(\theta_0,\alpha_{i0}) = -2\theta_0 \end{aligned}$$

Q4. Prove that

$$\frac{v_{it}}{E[v_{it}^{\alpha_i}]} = -w_{it}$$
$$U_{it}^{\alpha_i} - \frac{E[U_{it}^{\alpha_i\alpha_i}]}{2E[v_{it}^{\alpha_i}]} v_{it} = \varepsilon_{it}$$

Solution From Q1,

$$\frac{v_{it}}{E[v_{it}^{\alpha_i}]} = -v_{it} \equiv -v_{it}(\alpha_{i0}) = -(x_{it} - \alpha_{i0}) = -w_{it}$$

So

$$\begin{aligned}
U_{it}^{\alpha_i} - \frac{E[U_{it}^{\alpha_i\alpha_i}]}{2E[v_{it}^{\alpha_i}]} v_{it} &= U_{it}^{\alpha_i} + \frac{E[U_{it}^{\alpha_i\alpha_i}]}{2} w_{it} && \text{(see above)} \\
&= y_{it} - \theta_0\alpha_{i0} - \theta_0 w_{it} && (\because \text{Q3}) \\
&= y_{it} - \theta_0 x_{it} \\
&= \varepsilon_{it}
\end{aligned}$$

Q5. Prove that

$$\begin{aligned}
&-\left(-\frac{1}{n}\sum_{i=1}^{n} E[U_{it}^{\theta}]\right)^{-1} \left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{v_{it}}{E[v_{it}^{\alpha_i}]}\right)\left(\frac{1}{\sqrt{T}}\sum_{i=1}^{T}\left[U_{it}^{\alpha_i} - \frac{E[U_{it}^{\alpha_i\alpha_i}]}{2E[v_{it}^{\alpha_i}]} v_{it}\right]\right)\right] \\
&= \left(\frac{1}{n}\sum_{i=1}^{n}\alpha_{i0}^2\right)^{-1}\left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} w_{it}\right)\left(\frac{1}{\sqrt{T}}\sum_{i=1}^{T}\varepsilon_{it}\right)\right] \\
&\xrightarrow{p} \frac{\mathrm{Cov}(w_{it}, \varepsilon_{it})}{\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\alpha_{i0}^2}
\end{aligned}$$

Solution The first equality uses Q2 and Q4. Now assuming that $E[w_{it}] = E[\varepsilon_{it}] = 0$,

$$\begin{aligned}
\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} w_{it}\right)\left(\frac{1}{\sqrt{T}}\sum_{i=1}^{T}\varepsilon_{it}\right) &= \frac{1}{T}\sum_{s,t=1}^{T} w_{is}\varepsilon_{it} \\
&= \frac{1}{T}\sum_{t=1}^{T} w_{it}\varepsilon_{it} + \frac{1}{T}\sum_{s\neq t} w_{is}\varepsilon_{it} \\
&\xrightarrow{p} E[w_{it}\varepsilon_{it}] && (\because \text{iid}) \\
&= E\Big[(w_{it} - E[w_{it}])(\varepsilon_{it} - E[\varepsilon_{it}])\Big] \\
&= \mathrm{Cov}(w_{it}, \varepsilon_{it})
\end{aligned}$$

Assume that $\frac{1}{n}\sum_{i=1}^{n}\alpha_{i0}^2$ converges, then since $\mathrm{Cov}(w_{it}, \varepsilon_{it})$ is iid over i,

$$\left(\frac{1}{n}\sum_{i=1}^{n}\alpha_{i0}^2\right)^{-1}\left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} w_{it}\right)\left(\frac{1}{\sqrt{T}}\sum_{i=1}^{T}\varepsilon_{it}\right)\right] \xrightarrow{p} \frac{\mathrm{Cov}(w_{it}, \varepsilon_{it})}{\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\alpha_{i0}^2}$$