Sequential Estimation of Dynamic Discrete Games

Victor Aguirregabiria and Pedro Mira

Presented by
Yang, Yong Hyeon
DYNAMIC GAMES

• Variables
  – $x_t$: observable state variable
  – $\varepsilon_t$: unobservable state variable
  – $a_t$: observable decision variable (discrete, e.g. binary)

• Firms choose optimal $a_{it}$ that maximizes

$$V_i(x_t, \varepsilon_{it}) = \max_{a_{it}} E \left[ \beta^{s-t} \sum_{s=t}^{\infty} \tilde{\Pi}_i(a_s, x_s, \varepsilon_{is}) \middle| x_t, \varepsilon_{it} \right]$$

where expectations are over others’ strategies and $\varepsilon_{jt}$’s, meaning that
  – Firms’ profit depends on other firms’ characteristic and decision.
  – Firms observe their own private information before decision.
  – Firms’ private information affects their profit only.
DYNAMIC PROGRAMMING

• Usually we denote

\[ V_i(x_t, \varepsilon_{it}) = \max_{a_{it}} E \left[ \tilde{\Pi}_i(a_t, x_t, \varepsilon_{it}) + \beta V_i(x_{t+1}, \varepsilon_{it+1}) \bigg| x_t, \varepsilon_{it} \right] \]

or simply drop time subscripts to write

\[ V_i(x, \varepsilon_i) = \max_{a_i} E \left[ \tilde{\Pi}_i(a, x, \varepsilon_i) + \beta V_i(x', \varepsilon'_i) \bigg| x, \varepsilon_i \right] \]

• \( \tilde{\Pi} \) is known

• \( V \) is unique given \( \tilde{\Pi} \) and distribution of \( x' \) and \( \varepsilon'_i \).
ESTIMATION

• Firms’ decision depends on parameters.

• Estimate parameters that make firms’ decision seem reasonable!
  – GMM
    \[ E[a_i - P_i(x; \theta)|x] = 0 \]
    meaning that given \( x \), firm \( i \) chooses \( a_i = 1 \) with probability \( P_i(x; \theta) \).
  – MLE
    \[ \max \log L(\varepsilon; \theta) = \log L(a, x; \theta) \]

• Basically this requires us to solve for \( V_i \) explicitly.
  – \( P_i(x; \theta) \) can be derived from \( V_i(x, \varepsilon_i) \).
  – \( L(a, x; \theta) \) depends on likelihood of \( a \), which can be obtained from \( V_i(x, \varepsilon_i) \).
PROBLEM

• This task requires much time.
  – Need to numerically solve for $V_i$ using contraction mapping.
  – Need to simulate integrals (or use quadrature).
  – Need to repeat these for many candidate parameters.

\[
V_i(x, \varepsilon_i; \theta) = \max_{a_i} E \left[ \tilde{\Pi}_i(a, x, \varepsilon_i; \theta) + \beta V_i(x', \varepsilon'_i) \bigg| x, \varepsilon_i; \theta \right]
\]

• Still doable, but want to reduce time cost.
SOLUTION

• Try to bypass the step for contraction mapping and get $P_i(x)$ or $L(a, x; \theta)$ directly using data.
  – Hotz and Miller (1993): GMM in the context of individual dynamic decision
  – Hotz, Miller, Sanders and Smith (1994): include simulation
  – Pakes, Ostrovsky, and Berry (2004): extend to dynamic games
  – Aguirregabiria and Mira (2002): MLE, individual dynamics, iteration

• Solve for $V$ on some set of parameters and use it repeatedly
  – Ackerberg (2001): use importance sampling
SUMMARY OF THIS PAPER

• Extend Aguirregabiria and Mira (2002) to dynamic games
  – Need to consider strategic complexity

• Try to incorporate unobserved heterogeneity
  – Hotz and Miller cannot be applied to the case where unobserved heterogeneous variable exists.
    (to be explained later)

• Conclusion: Nested Pseudo Likelihood (NPL) estimator is useful when
  – multiple equilibria exist,
  – state spaces are large, and/or
  – conditional choice probabilities are not easily estimated
    (first and third will be explained later)
MODEL IN DETAIL

• Assume additive separability

\[
\tilde{\Pi}_i(a_t, x_t, \varepsilon_{it}) = \Pi_i(a_t, x_t) + \varepsilon_{it}(a_{it})
\]

• Assume conditional independence

\[
p(x_{t+1}, \varepsilon_{t+1} | a_t, x_t, \varepsilon_t) \\
= p(\varepsilon_{t+1} | x_{t+1}, a_t, x_t, \varepsilon_t) p(x_{t+1} | a_t, x_t, \varepsilon_t) \\
= p_{\varepsilon}(\varepsilon_{t+1}) f(x_{t+1} | a_t, x_t)
\]

• Assume independent error across players

• Assume discrete and finite \( x_t \)
VALUE FUNCTION

• Original value function is

\[
V_i(x, \varepsilon_i) = \max_{a_i} E \left[ \Pi_i(a, x) + \varepsilon_i(a_i) + \beta V_i(x', \varepsilon'_i) \mid x, \varepsilon_i \right]
\]

• Let \( \sigma = \{\sigma_i\}_{i=1}^N \) be strategy profiles players are using.

\[
V_i^\sigma(x, \varepsilon_i) = \max_{a_i} E \left[ \Pi_i^\sigma(a, x) + \varepsilon_i(a_i) + \beta V_i^\sigma(x', \varepsilon'_i) \mid x, \varepsilon_i \right]
\]

• Let \( \pi_i^\sigma(a_i, x) = E[\Pi_i^\sigma(a, x)] \) be the expected profit.

\[
V_i^\sigma(x, \varepsilon_i) = \max_{a_i} \left\{ \pi_i^\sigma(a_i, x) + \varepsilon_i(a_i) + \beta E[V_i^\sigma(x', \varepsilon'_i) \mid x, \varepsilon_i] \right\}
\]

• Explicitly writing integrals,

\[
V_i^\sigma(x, \varepsilon_i) = \max_{a_i} \left\{ \pi_i^\sigma(a_i, x) + \varepsilon_i(a_i)
\right.

+ \beta \sum_{x'} f^\sigma(x' \mid a_i, x) \int V_i^\sigma(x', \varepsilon'_i) g_i(\varepsilon'_i) \, d\varepsilon'_i \right\}
\]
CONDITIONAL CHOICE PROBABILITY

• Define $v_i^\sigma(a_i, x)$ as

$$v_i^\sigma(a_i, x) = \pi_i^\sigma(a_i, x) + \beta \sum_{x'} f_i^\sigma(x'|a_i, x) \int V_i^\sigma(x', \varepsilon'_i) g_i(\varepsilon'_i) d\varepsilon'_i$$

so that

$$V_i^\sigma(x, \varepsilon_i) = \max_{a_i} \left\{ v_i^\sigma(a_i, x) + \varepsilon_i(a_i) \right\}$$

• Conditional choice probability $P_i(x)$ is defined as

$$P_i(x) = E[a_i|x] = \Pr(a_i = 1|x) = \Pr \left( v_i^\sigma(1, x) + \varepsilon_i(1) > v_i^\sigma(0, x) + \varepsilon_i(0) \right)$$
INTEGRATED VALUE FUNCTION

• Use integrated value function to get $P_i(x)$.

$$V_i^\sigma(x) = \int V_i^\sigma(x, \varepsilon_i) g_i(\varepsilon_i) d\varepsilon_i$$

$$= \int \max_{a_i} \left\{ v_i^\sigma(a_i, x) + \varepsilon_i(a_i) \right\} g_i(\varepsilon_i) d\varepsilon_i$$

where now

$$v_i^\sigma(a_i, x) = \pi_i^\sigma(a_i, x) + \beta \sum_{x'} f^\sigma(x'|a_i, x) V_i^\sigma(x')$$

• Why is $V_i^\sigma(x)$ useful?

  – Decrease in the number of state spaces
  – Easy to calculate $v_i^\sigma(a_i, x)$: do not need to do integral
    → Easy to calculate $P_i(x)$: e.g. Type I extreme value distribution implies

$$P_i(x) = \frac{e^{v_i^\sigma(1,x)}}{e^{v_i^\sigma(0,x)} + e^{v_i^\sigma(1,x)}}$$
METHODOLOGY

• Need to do contraction mapping to get $V_i^\sigma(x)$?

$$V_i^\sigma(x) = \int \max_{a_i} \left\{ \pi_i^\sigma(a_i, x) + \varepsilon_i(a_i) + \beta \sum_{x'} f^\sigma(x'|a_i, x)V_i^\sigma(x') \right\} g_i(\varepsilon_i) d\varepsilon_i$$

• NO. Then we would have the same computational burden.

• Alternatively, if we have an estimate of $v_i^\sigma(a_i, x)$, say, $\hat{v}_i^\sigma(a_i, x)$,

$$\hat{V}_i^\sigma(x) = \int \max_{a_i} \left\{ \hat{v}_i^\sigma(a_i, x) + \varepsilon_i(a_i) \right\} g_i(\varepsilon_i) d\varepsilon_i$$

is a good estimate of $V_i^\sigma(x)$.

• Unfortunately, we do not have an estimate of $v_i^\sigma(a_i, x)$.

• But we can estimate $P_i(x)$. Can we use this?

• YES. It is actually a part of strategies $\sigma$. 
ALTERNATIVE WAY

- Find a way to go from $P_i(x)$ to $V_i^\sigma(x)$.

\[
V_i^\sigma(x) = \int \max_{a_i} \left\{ \pi_i^\sigma(a_i, x) + \varepsilon_i(a_i) + \beta \sum_{x'} f^\sigma(x'|a_i, x)V_i^\sigma(x') \right\} g_i(\varepsilon_i) d\varepsilon_i
\]

\[
= P_i(x) \left[ \pi_i^\sigma(1, x) + E[\varepsilon_i(1)|a_i = 1, x] + \beta \sum_{x'} f^\sigma(x'|a_i = 1, x)V_i^\sigma(x') \right] \\
+ \left[1 - P_i(x)\right] \left[ \pi_i^\sigma(0, x) + E[\varepsilon_i(0)|a_i = 0, x] + \beta \sum_{x'} f^\sigma(x'|a_i = 0, x)V_i^\sigma(x') \right]
\]

- Use estimated conditional choice probability $\widehat{P_i}(x)$ to get

\[
\widehat{V}_i^P(x) = \widehat{P_i}(x) \left[ \pi_i^P(1, x) + E[\varepsilon_i(1)|a_i = 1, x] + \beta \sum_{x'} \widehat{f^P}(x'|a_i = 1, x)\widehat{V}_i^P(x') \right] \\
+ \left[1 - \widehat{P_i}(x)\right] \left[ \pi_i^P(0, x) + E[\varepsilon_i(0)|a_i = 0, x] + \beta \sum_{x'} \widehat{f^P}(x'|a_i = 0, x)\widehat{V}_i^P(x') \right]
\]

- Since $x$ is finite, $\widehat{V}_i^P(x)$ can be easily obtained using matrix calculation.
SUMMARIZE THE PROCEDURE

(1) Estimate \( \hat{P}_i(x) \).

(2) Obtain \( \hat{\pi}_i^P(a_i, x; \theta) \) and \( \hat{f}^P(x'|a_i, x; \theta) \)

(3) Solve for \( \hat{V}_i^P(x; \theta) \)
   This is a good estimate of \( V_i^\sigma(x; \theta) \).

(4) Compute \( \hat{v}_i^P(a_i, x; \theta) \)
   \[
   \hat{v}_i^P(a_i, x; \theta) = \hat{\pi}_i^P(a_i, x; \theta) + \beta \sum_{x'} \hat{f}^P(x'|a_i, x; \theta) \hat{V}_i^P(x'; \theta)
   \]

(5) Derive \( \tilde{P}_i(x; \theta) \): e.g. Type I extreme value distribution implies
   \[
   \tilde{P}_i(x; \theta) = \frac{e^{\hat{v}_i^P(1,x;\theta)}}{e^{\hat{v}_i^P(0,x;\theta)} + e^{\hat{v}_i^P(1,x;\theta)}}
   \]

(6) (Hotz and Miller) Estimate parameters that satisfies
   \[
   E \left[ a_i - \tilde{P}_i(x; \theta) \bigg| x \right] = 0
   \]
THEORETICAL REPRESENTATION

- A Markov Perfect Equilibrium as a Fixed Point
  - Let $\Psi$ be a best response mapping.
  - Then, $P$, a set of conditional choice probabilities, is a MPE iff
    \[ P = \Psi(P) \]
    meaning a fixed point of $\Psi$.

- Put true $P$, then we would get $P$ in the last under true parameters.
INTRODUCTION TO ESTIMATION

• Hotz and Miller, and Pakes, Ostrovsky and Berry use GMM.
• Aguirregabiria and Mira use MLE.
  – But this is not a standard MLE.
  – Maximize likelihood of discrete (e.g. binary) choice
    \[
    \max_{\theta} \sum_{m=1}^{M} \sum_{t=1}^{T} \sum_{i=1}^{N} \log \tilde{P}_i(x_{mt})^{a_{imt}} \left[ 1 - \tilde{P}_i(x_{mt}) \right]^{1-a_{imt}}
    \]
    where \( m \) denotes markets.
  – Since \( \tilde{P}_i(x) \) is not a true probability when \( \theta \) is not a true parameter, we call this the pseudo likelihood function, and the maximizer is called the pseudo maximum likelihood (PML) estimator.

Note: The precise definition of pseudo likelihood function is broader

\[
Q_M(\theta, P) = \frac{1}{M} \sum_{m=1}^{M} \sum_{t=1}^{T} \sum_{i=1}^{N} \log \Psi_i(a_{imt}|x_{imt}; P, \theta)
\]

where \( \Psi_i \) is a best response mapping given the initial \( P \).
PML ESTIMATION

• Infeasible PML estimation
  
  – Suppose we know true $P$. Use this to get $\tilde{P}$.
  – Form a pseudo likelihood function, and find $\theta$ maximizing it.

  \[
  \sqrt{M} (\hat{\theta} - \theta) \xrightarrow{d} N(0, \Omega_{\theta\theta}^{-1})
  \]
  
  where $\Omega_{\theta\theta} \equiv E[\nabla_{\theta} s_m \nabla_{\theta} s'_m]$ is the information matrix.

• 2 step PML estimation

  – Estimate $\hat{P}$ nonparametrically so that $\sqrt{M}(\hat{P} - P) \xrightarrow{d} N(0, \Sigma)$.
  – Use this to get $\tilde{P}$.
  – Form a pseudo likelihood function, and find $\theta$ maximizing it.

  \[
  \sqrt{M} (\hat{\theta} - \theta) \xrightarrow{d} N(0, V_{2S})
  \]

  where $V_{2S} = \Omega_{\theta\theta}^{-1} + \Omega_{\theta\theta}^{-1} \Omega_{\theta P} \Sigma \Omega'_{\theta P} \Omega_{\theta\theta}^{-1}$

  – Identification assumptions are required.
IDENTIFICATION ASSUMPTION

- Denote true $P$ by $P^0$.
- Assume
  
  (A) For every observation $(m, t)$, $P^0$ is played.
  (B) Players expect $P^0$ to be played in future periods.
  (C) For any $\theta \neq \theta^0$, the solution of $P = \Psi(P)$ is $P \neq P^0$.
  (D) The observations are independent across markets.
DRAWBACK OF THE PML ESTIMATOR

• Inefficiency
  – $V_{2S}$ depends on $\Sigma$.
  – If $\hat{P}$ is inefficient, $\hat{\theta}$ is inefficient, too.

• Bias
  – Imprecise $\hat{P}$ makes huge bias, especially in small samples.
  – e.g. if we use crude frequency (accept/reject) estimator.

• Feasibility
  – $\hat{P}$ may not be estimated.
  – e.g. if unobserved heterogeneous variable exists.

• Motivates another estimator.
FIXED POINT OF THE PROCEDURE

• Iterations on Procedure
  
  – Start with $\hat{P}_0$.
  
  – Use this to get $\tilde{P}(x; \theta)$. Find $\hat{\theta}_1$ maximizing the PL function.
  
  – Define $\hat{P}_1(x) = \tilde{P}(x; \hat{\theta}_1)$.

  – Use this to get $\tilde{P}(x; \theta)$. Find $\hat{\theta}_2$ maximizing the PL function. ⋯

  – Define $\hat{P}_{K-1}(x) = \tilde{P}(x; \hat{\theta}_{K-1})$.

  – Use this to get $\tilde{P}(x; \theta)$. Find $\hat{\theta}_K$ maximizing the PL function.

• Includes 2 step PML ($K = 1$ with $\hat{P}_0 = \hat{P}$)

• As $K \to \infty$, the sequence $\{\hat{\theta}_K, \hat{P}_K\}$ converges.
  
  – Put its limit, then we get the same: a fixed point of the procedure.

  – If its limit is singleton, $\hat{\theta}_\infty$ is consistent.
NESTED PML ESTIMATION

• Nested PML (NPL) estimator is defined as follows.
  – Find all NPL fixed points.
  – Find the maximizer of PL function among them. Call it the NPL estimator.

• Properties
  – Consistent: \( \hat{\theta} \xrightarrow{p} \theta \)
  – Asymptotic Normality
    \[ \sqrt{M}(\hat{\theta} - \theta) \xrightarrow{d} N(0, V_{NPL}) \]
  – More efficient than the infeasible PML estimator under some condition.
  – Less finite sample bias.
  – Any initial \( \hat{P}_0 \) leads to the NPL estimator \( \hat{\theta} \).
    So can be applied to the case where \( \hat{P} \) may not be estimated.
  – Existence of multiple equilibria does not lead to multiple NPL fixed points.
  – Still require the same identification assumptions.
MONTE CARLO EXPERIMENT

• Use the model of firms’ entry and exit.
  – 5 firms choose to enter or exit upon 160 states.
  – Try to estimate cost and revenue parameters.

• Compare infeasible/crude frequency 2 step/logit 2 step PML estimators with the NPL estimator.
  – NPL always converges (faster with logit initial estimator).
  – It always converges to the same estimate regardless of initial guess.
  – Crude frequency 2 step PML has a very large bias.
  – NPL has sometimes less variance and MSE than infeasible PML.
  – Logit 2 step PML is pretty good.

• Using a smooth nonparametric estimator in the 1st step reduces bias in the 2nd step. (Pakes, Ostrovsky, and Berry)
PERMANENT UNOBSERVED HETEROGENEITY

- What if a state variable is unobserved and heterogeneous but permanent?
  - Let $\omega_m$ is a time-invariant market characteristic.
  - Common knowledge to the players but unobserved to the econometrician.
  - Need to construct $P_i(x, \omega)$ to use the same method.
  - Problem: cannot construct $P_i(x, \omega)$ since $\omega$ is not observed.

- Still doable with the original value function method.
- Infeasible with Hotz and Miller style estimation.
- Aguirregabiria and Mira propose the method and the condition.
CONDITION AND METHOD

• Assume conditional iid
  – Let $x_m$ be time invariant market characteristics.
  – Given $x_m$, $\omega_m$ is iid across markets: $\Pr(\omega_m = \omega^l | x_m) = \varphi_l(x_m)$
  – $\omega_m$ does not affect $f(x' | a, x)$

• The idea is that
  – write $\omega_m$ as a function of observable state variables and parameters.
  – residual is iid, so we can do ML type estimation.

\[
\log \Pr(\text{data}) = \sum_{m=1}^{M} \log \Pr(a_m, x_m)
= \sum_{m=1}^{M} \log \left( \sum_{l=1}^{L} \varphi_l(x_m) \Pr(a_m, x_m | \omega^l) \right)
\]
EMPIRICAL APPLICATION

• Use Chilean industries of five sectors.
  – Firms choose to enter or exit
  – There is a permanent unobserved heterogeneity $\omega_m$ taking on 21 values.

• Results
  – NPL always converges.
  – It always converges to the same estimate.
  – Unreasonable estimate without $\omega_m$: it says that a firm’s profit increases with the number of firms.
  – Including $\omega_m$ makes reasonable estimation.
  – Actual interpretation of parameters is not interesting.