**2003 Fall Part 1**

**1. Fisher Information**

$$\log f_X(x) = -\theta + x\log\theta - \log x!$$

$$\begin{aligned}\frac{\partial \log f_X(x)}{\partial\theta} &= -1 + \frac{x}{\theta}\\ \frac{\partial^2 \log f_X(x)}{\partial\theta^2} &= -\frac{x}{\theta^2}\end{aligned}$$

Therefore, Fisher Information is

$$I(\theta) = -E\left[\frac{\partial^2 \log f_X(x)}{\partial\theta^2}\right] = \frac{E[X]}{\theta^2} = \frac{1}{\theta}$$

**2. Asymptotic distribution**
Let $g(x) = x^2$, then $g(1) = 1$ and $g'(1) = 2$. Apply delta method,

$$\sqrt{n}\,(g(Z_n) - g(1)) \xrightarrow{d} N(0, [g'(1)]^2 \cdot 4)$$

$$\sqrt{n}\left(Z_n^2 - 1\right) \xrightarrow{d} N(0, 16)$$

**3. Summation of independent normal distriubtion**
MGF of $\sum_i a_i X_i$ is

$$\begin{aligned}E\left[e^{t\sum_i a_i X_i}\right] &= E\left[e^{ta_1X_1 + ta_2X_2 + \cdots + ta_nX_n}\right]\\ &= E\left[e^{ta_1X_1}e^{ta_2X_2}\cdots e^{ta_nX_n}\right]\\ &= E\left[e^{ta_1X_1}\right]E\left[e^{ta_2X_2}\right]\cdots E\left[e^{ta_nX_n}\right] \quad \because \text{independence}\\ &= \exp\left(\mu_1 ta_1 + \frac{1}{2}\sigma_1^2(ta_1)^2\right)\exp\left(\mu_2 ta_2 + \frac{1}{2}\sigma_2^2(ta_2)^2\right)\cdots\exp\left(\mu_n ta_n + \frac{1}{2}\sigma_n^2(ta_n)^2\right)\\ &= \exp\left([a_1\mu_1 + a_2\mu_2 + \cdots a_n\mu_n]t + \frac{1}{2}[a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \cdots + a_n^2\sigma_n^2]t^2\right)\end{aligned}$$

This is MGF of normal distribution with mean $\sum_i a_i\mu_i$ and variance $\sum_i a_i^2\sigma_i^2$.

**4. Method of Moments (a)**

$$E[X_i^2] = \text{var}(X_i) + (E[X_i])^2 = \theta + \theta^2$$

**(b)**
Substitute moment condition with the sample moment condition

$$\begin{aligned}\frac{1}{100}\sum_{i=1}^{100} X_i^2 &= \theta + \theta^2\\ 6 &= \theta + \theta^2\end{aligned}$$

Solving the equation, $\theta = 2$ or $-3$. Since $\theta = E[X_i] = \text{var}(X_i) > 0$,

$$\widehat{\theta}_{MM} = 2$$