2003 Spring Part 1

1. Fisher Information

$$\log f(x) = -\frac{1}{2}\log 2\pi - \frac{1}{2}\log\theta_2 - \frac{(x-\theta_1)^2}{2\theta_2}$$

$$\frac{\partial \log f(x)}{\partial\theta} = \begin{pmatrix} \frac{1}{\theta_2}(x-\theta_1) \\ -\frac{1}{2\theta_2} + \frac{1}{2\theta_2^2(x-\theta_1)^2} \end{pmatrix}$$

Fisher Information is

$$I(\theta) = E\left[\frac{\partial \log f(x)}{\partial\theta}\frac{\partial \log f(x)}{\partial\theta'}\right] \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where

$$\begin{aligned} A &= E\left[\frac{(X-\theta_1)^2}{\theta_2^2}\right] = \frac{\theta_2}{\theta_2^2} = \frac{1}{\theta_2} \\ B = C &= E\left[-\frac{1}{2\theta_2^2}(X-\theta_1) + \frac{1}{2\theta_2^3}(X-\theta_1)^3\right] = 0 \\ D &= E\left[\frac{1}{4\theta_2^2} - \frac{1}{2\theta_2^3}(X-\theta_1)^2 + \frac{1}{4\theta_2^4}(X-\theta)^4\right] = \frac{1}{4\theta_2^2} - \frac{\theta_2}{2\theta_2^3} + \frac{3\theta_2^2}{4\theta_2^4} = \frac{1}{2\theta_2^2} \end{aligned}$$

2. Asymptotic variance of MLE (a)

Apply the asymptotic efficiency theorem of MLE $\sqrt{n}(\widehat{\theta}_{MLE} - \theta) \xrightarrow{d} N\left(0, I(\theta)^{-1}\right)$

$$\log f(x,\theta) = \log\theta + (\theta - 1)\log x$$

$$\begin{aligned} \frac{\partial \log f(x,\theta)}{\partial\theta} &= \frac{1}{\theta} + 1 \\ \frac{\partial^2 \log f(x,\theta)}{\partial\theta^2} &= -\frac{1}{\theta^2} \end{aligned}$$

Fisher Information is

$$I(\theta) = -E\left[\frac{\partial^2 \log f(X,\theta)}{\partial\theta^2}\right] = \frac{1}{\theta^2}$$

The asymptotic variance of $\widehat{\theta}_{MLE}$ is $I(\theta)^{-1} = \theta^2$.

(b)

$$\log f(x,\theta) = -\log\theta + \frac{x}{\theta}$$

$$\begin{aligned} \frac{\partial \log f(x,\theta)}{\partial\theta} &= -\frac{1}{\theta} + \frac{x}{\theta^2} \\ \frac{\partial^2 \log f(x,\theta)}{\partial\theta^2} &= \frac{1}{\theta^2} - \frac{2x}{\theta^3} \end{aligned}$$

Fisher Information is

$$I(\theta) = -E\left[\frac{\partial^2 \log f(X,\theta)}{\partial\theta^2}\right] = -\frac{1}{\theta^2} + \frac{2E[X]}{\theta^3} = \frac{1}{\theta^2}$$

where we used $E[X] = \theta$ since $X \sim \Gamma(1,\theta)$. The asymptotic variance of $\widehat{\theta}_{MLE}$ is $I(\theta)^{-1} = \theta^2$.

3. Optimal test

From Neyman-Pearson theorem, the optimal critical region is given by

$$C = \left\{ x = (x_1, \cdots, x_n) \left| \frac{\prod_{i=1}^{25} f(x, \mu_1)}{\prod_{i=1}^{25} f(x, \mu_0)} \geq k \right. \right\} = \left\{ x \left| \frac{1}{25} \sum_{i=1}^{25} x_i \geq c \right. \right\}$$

Find c such that $\Pr(X \in C) = 0.05$ under H_0. Since $\frac{1}{25}\sum_{i=1}^{25} X_i \sim N\left(\mu_0, \frac{1}{25}\right)$,

$$0.05 = \Pr\left(\frac{\left[\frac{1}{25}\sum_{i=1}^{25} X_i - \mu_0\right]}{\sqrt{1/25}} \geq 1.645 \right) = \Pr\left(\frac{1}{25}\sum_{i=1}^{25} X_i \geq \mu_0 + 0.329 \right)$$

Therefore, the optimal critical region is

$$C = \left\{ x \left| \frac{1}{25} \sum_{i=1}^{25} x_i \geq \mu_0 + 0.329 \right. \right\}$$

4. Asypmtotic distriubtion

When $n = 4$,

$$S \geq \frac{1.96}{\sqrt{n}} \cdot \frac{1}{2} \Longleftrightarrow \frac{1}{4}\sum_{i=1}^{4} 1(X_i \geq 0) \geq 0.99 \Longleftrightarrow X_i \geq 0 \quad \forall i = 1, 2, 3, 4$$

Under H_0,

$$\Pr(X_i \geq 0\ \forall i) = \prod_{i=1}^{4} \Pr(X_i \geq 0) = \frac{1}{16} \quad \because iid$$

When $n \to \infty$,

$$S \geq \frac{1.96}{\sqrt{n}} \cdot \frac{1}{2} \Longleftrightarrow \sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n} 1(X_i \geq 0) - \frac{1}{2}\right] \geq \frac{1}{2} \times 1.96$$

Under H_0, $1(X_i \geq 0)$ has a Bernoulli distribution $B\left(\frac{1}{2}\right)$. Thus

$$\begin{aligned} E\left[1(X_i \geq 0)\right] &= \frac{1}{2} \\ \text{var}\left[1(X_i \geq 0)\right] &= \frac{1}{4} \end{aligned}$$

Apply CLT,

$$\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n} 1(X_i \geq 0) - \frac{1}{2}\right] \xrightarrow{d} N\left(0, \frac{1}{4}\right)$$

Therefore,

$$\lim_{n\to\infty} \Pr\left(S \geq \frac{1.96}{\sqrt{n}} \cdot \frac{1}{2}\right) = \lim_{n\to\infty} \Pr\left(\frac{\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n} 1(X_i \geq 0) - \frac{1}{2}\right]}{\frac{1}{2}} \geq 1.96 \right) = 0.025$$