2004 Fall Part 1

1. Uniformly most powerful test (1)

$$\Pr(\overline{X} \le 1 | H_0) = \Pr\left(\sum_{i=1}^{n} \le 5 | H_0\right) = 1$$

(2)
To get 100% significance level, we have to reject all the time!

2. Confidence interval

$$\overline{U} = \frac{1}{64}\sum_{i=1}^{64} U_i \sim N\left(\mu, \frac{1}{64}\right)$$

When $\overline{U} = 3$ is observed,

$$\begin{aligned} 0.95 &= \Pr\left(\frac{|\overline{U} - \mu|}{\sqrt{1/64}} \le 1.96\right) \\ &= \Pr\left(|\overline{U} - \mu| \le 0.245\right) \\ &= \Pr\left(2.755 \le \mu \le 3.245\right) \end{aligned}$$

Thus CI is [2.755,3.245]

3. Minimal variance
Verify that

$$E[\overline{X}] = \frac{1}{n}\sum_{i=1}^{n} EX_i = \theta$$

Let Y be any unbiased estimator of θ, then by Cramer-Rao Lower Bound, $\mathrm{var}(Y) \ge I(\theta)^{-1}$. Log likelihood is

$$\mathcal{L} = -\frac{n}{2}\log 2\pi\sigma^2 - \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \theta)^2$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \theta} &= \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \theta) \\ \frac{\partial^2 \mathcal{L}}{\partial \theta^2} &= -\frac{n}{\sigma^2} \end{aligned}$$

Fisher Information is

$$I(\theta) = E\left[\frac{\partial^2 \mathcal{L}}{\partial \theta^2}\right] = \frac{n}{\sigma^2}$$

Confirm that $\overline{X}$ attains this variance $I(\theta)^{-1}$, that is,

$$\mathrm{var}(\overline{X}) = \frac{1}{n}\mathrm{var}(X_i) = \frac{\sigma^2}{n}$$

4. Moment generating function
Let $M(t) = \exp(\mu[e^t - 1])$. Note that $M'(t) = \mu e^t M(t)$ and $M''(t) = \mu e^t M(t) + (\mu e^t)^2 M(t)$.

$$E[X] = M'(0) = \mu$$

$$\mathrm{var}(X) = E[X^2] - (E[X])^2 = M''(0) - \mu^2 = \mu$$