2004 Spring Part 1

1. Transformation of normal distribution
Define A as

$$A = \begin{pmatrix} 1 & 0 \\ \frac{-\rho\sigma_2}{\sigma_1} & 1 \end{pmatrix}$$

then

$$AX \sim N\left(0, A\begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} A'\right)$$

Here

$$\begin{aligned} AX &= \begin{pmatrix} X_1 \\ X_2 - \frac{\rho\sigma_2}{\sigma_1}X_1 \end{pmatrix} \\ A\begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} A' &= \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ 0\sigma_2 & (1-\rho^2)\sigma_2^2 \end{pmatrix} A' = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & (1-\rho^2)\sigma_2^2 \end{pmatrix} \end{aligned}$$

Since X_1 and $U = X_2 - \frac{\rho\sigma_2}{\sigma_1}X_1$ are jointly normal and $\text{cov}(X_1, U) = 0$, they are independent of each other.

2. Uniformly most powerful critical region
A uniformly most powerful critical region is given by

$$C = \left\{ x = (x_1, \cdots, x_n) \middle| \frac{\prod_{i=1}^n f(x_i, \theta'')}{\prod_{i=1}^n f(x_i, \theta')} \geq k \quad \forall \theta'' < \theta' \right\}$$

Remember that

$$\begin{aligned} \prod_{i=1}^n f(x_i, \theta'') &= (2\pi\theta'')^{-\frac{n}{2}} \exp\left(-\frac{\sum_{i=1}^n x_i^2}{2\theta''}\right) \\ \prod_{i=1}^n f(x_i, \theta') &= (2\pi\theta')^{-\frac{n}{2}} \exp\left(-\frac{\sum_{i=1}^n x_i^2}{2\theta'}\right) \end{aligned}$$

Thus

$$\begin{aligned} \frac{\prod_{i=1}^n f(x_i, \theta'')}{\prod_{i=1}^n f(x_i, \theta')} \geq k &\iff \left(\frac{\theta'}{\theta''}\right)^{\frac{n}{2}} \exp\left(\left[\frac{1}{2\theta'} - \frac{1}{2\theta''}\right] \sum_{i=1}^n x_i^2\right) \geq k \\ &\iff \left(\frac{1}{2\theta'} - \frac{1}{2\theta''}\right) \sum_{i=1}^n x_i^2 \geq \log\left[k\left(\frac{\theta''}{\theta'}\right)^{\frac{n}{2}}\right] \\ &\iff \sum_{i=1}^n x_i^2 \leq \frac{2\theta'\theta''}{\theta''-\theta'} \log\left[k\left(\frac{\theta''}{\theta'}\right)^{\frac{n}{2}}\right] \equiv c \end{aligned}$$

Noe that $c > 0$ since $\theta'' < \theta'$. The above result holds for any $\theta'' < \theta'$, so $\{x | \sum_{i=1}^n x_i^2 \leq c\}$ defines a uniformly most powerful critical region.

3. MLE
Use Invariance principle. MLE of θ is obtained by

$$\max \sum_{i=1}^n \log f(x) = \sum_{i=1}^n [x_i \log\theta - \theta - \log x!]$$

FOC is

$$\frac{\sum_{i=1}^n x_i}{\theta} - n = 0$$

$$\widehat{\theta}_{MLE} = \frac{1}{n}\sum_{i=1}^{n} x_i = \overline{x}$$

By Invariance principle, MLE of $\Pr[X_{10} = 1 \text{ or } 2]$ is

$$\widehat{\Pr}[X_{10} = 1 \text{ or } 2] = \frac{\overline{x}e^{-\overline{x}}}{1!} + \frac{\overline{x}^2 e^{-\overline{x}}}{2!} = e^{-\overline{x}}\left(\overline{x} + \frac{1}{2}\overline{x}^2\right)$$

4. Minimal variance

Let $f(x) = \pi^x(1-\pi)^{1-x}$ be the pdf of X_i. Verify that

$$E[\overline{X}] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \pi$$

By Cramer-Rao Lower Bound, any unbiased estimator Y of π has

$$\text{var}(Y) \geq I(\pi)^{-1}$$

Log-likelihood is

$$\mathcal{L} = \sum_{i=1}^{n} \log f(x) = \sum_{i=1}^{n} x_i \log \pi + (n - \sum_{i=1}^{n} x_i)\log(1-\pi)$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \pi} &= \frac{\sum_{i=1}^{n} x_i}{\pi} - \frac{n - \sum_{i=1}^{n} x_i}{1-\pi} \\ \frac{\partial^2 \mathcal{L}}{\partial \pi^2} &= -\frac{\sum_{i=1}^{n} x_i}{\pi^2} - \frac{n - \sum_{i=1}^{n} x_i}{(1-\pi)^2} \end{aligned}$$

Fisher Information is

$$I(\pi) = -E\left[\frac{\partial^2 \mathcal{L}}{\partial \pi^2}\right] = \frac{n\pi}{\pi^2} + \frac{n - n\pi}{(1-\pi)^2} = \frac{n}{\pi(1-\pi)}$$

Now verify that $\overline{X}$ attains this variance $I(\pi)^{-1}$.

$$\text{var}(\overline{X}) = \frac{1}{n}\text{var}(X_i) = \frac{\pi(1-\pi)}{n} \quad \because iid$$