1. Transformation of normal distribution
Define \( A \) as
\[
A = \begin{pmatrix}
\frac{1}{\rho \sigma_2} & 0 \\
-\rho \sigma_2 & \sigma_1
\end{pmatrix}
\]
then
\[
AX \sim N \left( 0, A \left( \begin{array}{cc}
\sigma_1^2 & \rho \sigma_1 \sigma_2 \\
\rho \sigma_1 \sigma_2 & \sigma_2^2
\end{array} \right) A' \right)
\]
Here
\[
AX = \begin{pmatrix}
X_1 \\
X_2 \frac{\rho \sigma_2}{\sigma_1} X_1
\end{pmatrix}
\begin{pmatrix}
\sigma_1^2 & \rho \sigma_1 \sigma_2 \\
\rho \sigma_1 \sigma_2 & \sigma_2^2
\end{pmatrix}
\begin{pmatrix}
X_1 \\
X_2 \frac{\rho \sigma_2}{\sigma_1} X_1
\end{pmatrix}
\begin{pmatrix}
\sigma_1^2 & 0 \\
0 & (1 - \rho^2) \sigma_2^2
\end{pmatrix}
\]
Since \( X_1 \) and \( U = X_2 \frac{\rho \sigma_2}{\sigma_1} X_1 \) are jointly normal and \( \text{cov}(X_1, U) = 0 \), they are independent of each other.

2. Uniformly most powerful critical region
A uniformly most powerful critical region is given by
\[
C = \left\{ x = (x_1, \ldots, x_n) \left| \prod_{i=1}^{n} f(x_i, \theta'') \prod_{i=1}^{n} f(x_i, \theta' \geq k \quad \forall \theta'' < \theta' \right. \right\}
\]
Remember that
\[
\prod_{i=1}^{n} f(x_i, \theta'') = (2\pi \theta'')^{-\frac{n}{2}} \exp \left( -\sum_{i=1}^{n} \frac{x_i^2}{2\theta''} \right)
\]
\[
\prod_{i=1}^{n} f(x_i, \theta') = (2\pi \theta')^{-\frac{n}{2}} \exp \left( -\sum_{i=1}^{n} \frac{x_i^2}{2\theta'} \right)
\]
Thus
\[
\frac{\prod_{i=1}^{n} f(x_i, \theta'')}{\prod_{i=1}^{n} f(x_i, \theta')} \geq k \iff \left( \frac{\theta'}{\theta''} \right)^{\frac{n}{2}} \exp \left( \left( \frac{1}{2\theta'} - \frac{1}{2\theta''} \right) \sum_{i=1}^{n} x_i^2 \right) \geq k
\]
\[
\iff \left( \frac{1}{2\theta'} - \frac{1}{2\theta''} \right) \sum_{i=1}^{n} x_i^2 \geq \log \left[ k \left( \frac{\theta''}{\theta'} \right)^{\frac{n}{2}} \right]
\]
\[
\iff \sum_{i=1}^{n} x_i^2 \leq \frac{2\theta' \theta''}{\theta'' - \theta'} \log \left[ k \left( \frac{\theta''}{\theta'} \right)^{\frac{n}{2}} \right] \equiv c
\]
Noe that \( c > 0 \) since \( \theta'' < \theta' \). The above result holds for any \( \theta'' < \theta' \), so \( \{ x | \sum_{i=1}^{n} x_i^2 \leq c \} \) defines a uniformly most powerful critical region.

3. MLE
Use Invariance principle. MLE of \( \theta \) is obtained by
\[
\max \sum_{i=1}^{n} \log f(x) = \sum_{i=1}^{n} \left[ x_i \log \theta - \theta - \log x_i! \right]
\]
FOC is
\[
\frac{\sum_{i=1}^{n} x_i}{\theta} - n = 0
\]
\[ \hat{\theta}_{MLE} = \frac{1}{n} \sum_{i=1}^{n} x_i = \bar{x} \]

By Invariance principle, MLE of \( \Pr[X_{10} = 1 \text{ or } 2] \) is
\[ \hat{\Pr}[X_{10} = 1 \text{ or } 2] = \frac{\pi e^{-\bar{x}}}{1!} + \frac{\pi^2 e^{-\bar{x}}}{2!} = e^{-\bar{x}} \left( \pi + \frac{1}{2} \pi^2 \right) \]

4. Minimal variance
Let \( f(x) = \pi^x (1 - \pi)^{1-x} \) be the pdf of \( X_i \). Verify that
\[ E[X] = \frac{1}{n} \sum_{i=1}^{n} E[X_i] = \pi \]

By Cramer-Rao Lower Bound, any unbiased estimator \( Y \) of \( \pi \) has
\[ \text{var}(Y) \geq I(\pi)^{-1} \]

Log-likelihood is
\[ \mathcal{L} = \sum_{i=1}^{n} \log f(x) = \sum_{i=1}^{n} x_i \log \pi + (n - \sum_{i=1}^{n} x_i) \log(1 - \pi) \]
\[ \frac{\partial \mathcal{L}}{\partial \pi} = \frac{\sum_{i=1}^{n} x_i}{\pi} - \frac{n - \sum_{i=1}^{n} x_i}{1 - \pi} \]
\[ \frac{\partial^2 \mathcal{L}}{\partial \pi^2} = -\frac{\sum_{i=1}^{n} x_i}{\pi^2} - \frac{n - \sum_{i=1}^{n} x_i}{(1 - \pi)^2} \]

Fisher Information is
\[ I(\pi) = -E \left[ \frac{\partial^2 \mathcal{L}}{\partial \pi^2} \right] = \frac{n\pi}{\pi^2} + \frac{n - n\pi}{(1 - \pi)^2} = \frac{n}{\pi(1 - \pi)} \]

Now verify that \( \bar{X} \) attains this variance \( I(\pi)^{-1} \).
\[ \text{var}(\bar{X}) = \frac{1}{n} \text{var}(X_i) = \frac{\pi(1 - \pi)}{n} \quad \because iid \]