2005 Fall Part 1

1. Confidence interval with t-test (a)
Since $\widehat{\beta}-\beta=(X'X)^{-1}X'\varepsilon$ and $\varepsilon\sim N(0,\sigma^2 I_n)$,

$$\widehat{\beta}-\beta\sim N\left(0,(X'X)^{-1}X'\sigma^2 I_n X(X'X)^{-1}\right)=N\left(0,\sigma^2(X'X)^{-1}\right)$$

(b)
It suffices to show that $\widehat{\beta}-\beta$ and e are independent of each other.

$$\begin{aligned}
e &= y-X\beta \\
&= y-X(X'X)^{-1}X'y \\
&= \left[I-X(X'X)^{-1}X'\right]y \\
&= \left[I-X(X'X)^{-1}X'\right](X\beta+\varepsilon) \\
&= \left[I-X(X'X)^{-1}X'\right]\varepsilon
\end{aligned}$$

Recall that when Z is normal, AZ and BZ are indendent of each other iff $AB'=0$. Since

$$\left[I-X(X'X)^{-1}X'\right]X(X'X)^{-1}=X(X'X)^{-1}-X(X'X)^{-1}=0$$

$\widehat{\beta}-\beta$ and e are independent of each other.

(c)
Recall that when $Z\sim N(0,I_n)$ and P is symmetric and idempotent, we have

$$Z'PZ\sim\chi^2_{tr(P)}$$

Let $P=\left[I-X(X'X)^{-1}X'\right]$, then P is symmetric and idempotent. Since $e=P\varepsilon$,

$$e'e=(P\varepsilon)'(P\varepsilon)=\varepsilon'P\varepsilon$$

Thus

$$\frac{e'e}{\sigma^2}=\left(\frac{\varepsilon}{\sigma}\right)'P\left(\frac{\varepsilon}{\sigma}\right)$$

Here $\left(\frac{\varepsilon}{\sigma}\right)\sim N(0,I_n)$, so we only need to show that $tr(P)=n-k$.

$$\begin{aligned}
tr(P) &= tr(I_n)-tr\left(X(X'X)^{-1}X'\right) \\
&= n-tr\left((X'X)^{-1}X'X\right) \\
&= n-tr(I_k) \\
&= n-k
\end{aligned}$$

(d)
Recall that when $Z\sim N(0,1)$ and $Y\sim\chi^2_p$ are independent of each other,

$$\frac{Z}{\sqrt{Y/p}}\sim t(p)$$

From (a),

$$c'(\widehat{\beta}-\beta)\sim N\left(0,\sigma^2c'(X'X)^{-1}c\right)$$

and thus

$$z\equiv\frac{c'(\widehat{\beta}-\beta)}{\sqrt{\sigma^2c'(X'X)^{-1}c}}\sim N(0,1)$$

From (c),

$$y \equiv \frac{(n-k)s^2}{\sigma^2} \sim \chi^2_{n-k}$$

From (b), z and y are independent of each other. Therefore

$$\frac{z}{\sqrt{y/(n-k)}} = \frac{c'(\widehat{\beta}-\beta)/\sqrt{\sigma^2 c'(X'X)^{-1}c}}{\sqrt{s^2/\sigma^2}} = \frac{c'(\widehat{\beta}-\beta)}{\sqrt{s^2 c'(X'X)^{-1}c}} \sim t(n-k)$$

So

$$\Pr\left(\left|\frac{c'(\widehat{\beta}-\beta)}{\sqrt{s^2 c'(X'X)^{-1}c}}\right| \leq t_{.975,n-k}\right) = 0.95$$

and 95% confidence interval is

$$c'\widehat{\beta} \pm t_{.975,n-k}\sqrt{s^2 c'(X'X)^{-1}c}$$

2. Consistency

By Markov inequality,

$$\Pr\left[|Y_n - \theta| \geq \varepsilon\right] \leq \frac{E\left[|Y_n-\theta|^2\right]}{\varepsilon^2}$$

Note that

$$E\left[|Y_n-\theta|^2\right] = E\left[|Y_n - E[Y_n] + E[Y_n] - \theta|^2\right] = E\left[|Y_n - E[Y_n]|^2\right] + \left(E[Y_n]-\theta\right)^2$$

Here

$$\begin{aligned}\lim_{n\to\infty} E\left[|Y_n - E[Y_n]|^2\right] &= \lim_{n\to\infty} \text{var}(Y_n) = 0 \\ \lim_{n\to\infty} \left(E[Y_n]-\theta\right)^2 &= \left(\lim_{n\to\infty} E[Y_n] - \theta\right)^2 = 0\end{aligned}$$

Therefore

$$\lim_{n\to\infty} \Pr\left[|Y_n-\theta| \geq \varepsilon\right] \leq \lim_{n\to\infty} \frac{E\left[|Y_n - E[Y_n]|^2\right] + \left(E[Y_n]-\theta\right)^2}{\varepsilon^2} = 0$$

for any $\varepsilon > 0$, which is equivalent to $Y_n \xrightarrow{p} \theta$.

3. Summation of independent Poisson distribution

MGF of $\sum_{i=1}^n X_i$ is

$$\begin{aligned}E\left[e^{t\sum_{i=1}^n X_i}\right] &= E\left[e^{tX_1+tX_2+\cdots+tX_n}\right] \\ &= E\left[e^{tX_1}e^{tX_2}\cdots e^{tX_n}\right] \\ &= E\left[e^{tX_1}\right]E\left[e^{tX_2}\right]\cdots E\left[e^{tX_n}\right] \quad \because \text{independence} \\ &= \exp\left(\mu_1[e^t-1]\right)\exp\left(\mu_2[e^t-1]\right)\cdots\exp\left(\mu_n[e^t-1]\right) \\ &= \exp\left([\mu_1+\mu_2+\cdots+\mu_n][e^t-1]\right)\end{aligned}$$

This is MGF of Poisson distribution with mean $\sum_{i=1}^n \mu_i$.

4. Convergence

($\Rightarrow$)

$$\lim_{n\to\infty} \Pr(Y_n \leq k) \begin{cases} \leq \lim_{n\to\infty} 1 - \Pr(|Y_n - c| \leq c-k) = 0 & \text{when } k < c \\ \geq \lim_{n\to\infty} \Pr(|Y_n - c| \leq k-c) = 1 & \text{when } k > c \end{cases}$$

Therefore the cdf of Y_n converges to the cdf of constant c, which means that $Y_n \xrightarrow{d} c$.

($\Leftarrow$)

$$\lim_{n\to\infty} \Pr(|Y_n - c| \leq \varepsilon) = \lim_{n\to\infty} F_{Y_n}(c+\varepsilon) - F_{Y_n}(c-\varepsilon) = 1$$

for any $\varepsilon > 0$, so $Y_n \xrightarrow{p} c$.