2005 Spring Part 1

1. Convergence

MGF of Y_n is

$$M(t) = \left(1 - \frac{\pi}{n} + \frac{\pi}{n}e^t\right)^n = \left(1 + \frac{\pi}{n}[e^t - 1]\right)^n$$

Recall that

$$\lim_{n\to\infty}\left(1 + \frac{b}{n}\right)^{cn} = e^{bc}$$

Since

$$\lim_{n\to\infty} M(t) = \lim_{n\to\infty}\left(1 + \frac{\pi}{n}[e^t - 1]\right)^n = \exp\left(\pi[e^t - 1]\right)$$

Y_n converges in distirubtion to Poisson distribution with mean π.

2. Asymptotic distribution (a)

Since $X_i = \sigma Z_i + \mu$ and $\overline{X} = \sigma\overline{Z} + \mu$,

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^n (X_i - \overline{X})^2 &= \frac{1}{n}\sum_{i=1}^n (\sigma Z_i - \sigma\overline{Z})^2 = \sigma^2\frac{1}{n}\sum_{i=1}^n (Z_i - \overline{Z})^2 \\
\frac{1}{n}\sum_{i=1}^n (X_i - \overline{X})^4 &= \frac{1}{n}\sum_{i=1}^n (\sigma Z_i - \sigma\overline{Z})^4 = \sigma^4\frac{1}{n}\sum_{i=1}^n (Z_i - \overline{Z})^4
\end{aligned}$$

Thus

$$Y_n = \frac{\sigma^4\frac{1}{n}\sum_{i=1}^n (Z_i - \overline{Z})^4}{\sigma^4\left[\frac{1}{n}\sum_{i=1}^n (Z_i - \overline{Z})^2\right]^2} - 3 = \frac{\frac{1}{n}\sum_{i=1}^n (Z_i - \overline{Z})^4 - 3\left[\frac{1}{n}\sum_{i=1}^n (Z_i - \overline{Z})^2\right]^2}{\left[\frac{1}{n}\sum_{i=1}^n (Z_i - \overline{Z})^2\right]^2}$$

(b)

By LLN,

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^n Z_i &\xrightarrow{p} E[z_i] = 0 \\
\frac{1}{n}\sum_{i=1}^n Z_i^2 &\xrightarrow{p} E[z_i^2] = 1
\end{aligned}$$

Therefore, by continuity,

$$\frac{1}{n}\sum_{i=1}^n (Z_i - \overline{Z})^2 = \frac{1}{n}\sum_{i=1}^n Z_i^2 - \left[\frac{1}{n}\sum_{i=1}^n Z_i\right]^2 \xrightarrow{p} 1 - 0 = 1$$

$$\left[\frac{1}{n}\sum_{i=1}^n (Z_i - \overline{Z})^2\right]^2 \xrightarrow{p} 1$$

(c)

By CLT,

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^n \begin{pmatrix} z_i^4 \\ z_i^3 \\ z_i^2 \\ z_i \end{pmatrix} - E\left[\begin{pmatrix} z_i^4 \\ z_i^3 \\ z_i^2 \\ z_i \end{pmatrix}\right]\right) \xrightarrow{d} N\left(0, \begin{pmatrix} \mathrm{var}(z_i^4) & \mathrm{cov}(z_i^4, z_i^3) & \mathrm{cov}(z_i^4, z_i^2) & \mathrm{cov}(z_i^4, z_i) \\ \mathrm{cov}(z_i^3, z_i^4) & \mathrm{var}(z_i^3) & \mathrm{cov}(z_i^3, z_i^2) & \mathrm{cov}(z_i^3, z_i) \\ \mathrm{cov}(z_i^2, z_i^4) & \mathrm{cov}(z_i^2, z_i^3) & \mathrm{var}(z_i^2) & \mathrm{cov}(z_i^2, z_i) \\ \mathrm{cov}(z_i^1, z_i^4) & \mathrm{cov}(z_i^1, z_i^3) & \mathrm{cov}(z_i^3, z_i^2) & \mathrm{var}(z_i) \end{pmatrix}\right)$$

Here we have

$$
\begin{aligned}
\mathrm{var}(z_i^4) &= E[z_i^8] - \big(E[z_i^4]\big)^2 = 105 - 9 = 96 \\
\mathrm{var}(z_i^3) &= E[z_i^6] - \big(E[z_i^3]\big)^2 = 15 - 0 = 15 \\
\mathrm{var}(z_i^2) &= E[z_i^4] - \big(E[z_i^2]\big)^2 = 3 - 1 = 2 \\
\mathrm{var}(z_i^1) &= E[z_i^2] - \big(E[z_i^1]\big)^2 = 1 - 0 = 1 \\
\mathrm{cov}(z_i^4, z_i^3) &= E[z_i^7] - E[z_i^4]E[z_i^3] = 0 \\
\mathrm{cov}(z_i^4, z_i^2) &= E[z_i^6] - E[z_i^4]E[z_i^2] = 15 - 3 \times 1 = 12 \\
\mathrm{cov}(z_i^4, z_i) &= E[z_i^5] - E[z_i^4]E[z_i] = 0 \\
\mathrm{cov}(z_i^3, z_i^2) &= E[z_i^5] - E[z_i^3]E[z_i^2] = 0 \\
\mathrm{cov}(z_i^3, z_i) &= E[z_i^4] - E[z_i^3]E[z_i] = 3 - 0 = 3 \\
\mathrm{cov}(z_i^2, z_i) &= E[z_i^3] - E[z_i^2]E[z_i] = 0
\end{aligned}
$$

and thus the desired result follows.

(d)
Use delta method. Define g as

$$
g\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = a - 4db + 12d^2c - 3c^2 - 6d^4
$$

Then

$$
\frac{dg}{d\begin{pmatrix} a & b & c & d \end{pmatrix}} = \begin{pmatrix} 1, & -4d, & 12d^2 - 6c, & -4b + 24dc - 24d^3 \end{pmatrix}
$$

Verify that

$$
g\begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} = 0 \quad , \quad \frac{dg}{d\begin{pmatrix} 3 & 0 & 1 & 0 \end{pmatrix}} = \begin{pmatrix} 1 & 0 & -6 & 0 \end{pmatrix}
$$

Thus we have

$$
\begin{aligned}
\sqrt{n}\left(g\begin{pmatrix} \frac{1}{n}\sum_{i=1}^n z_i^4 \\ \frac{1}{n}\sum_{i=1}^n z_i^3 \\ \frac{1}{n}\sum_{i=1}^n z_i^2 \\ \frac{1}{n}\sum_{i=1}^n z_i \end{pmatrix} - g\begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right) &\xrightarrow{d} \begin{pmatrix} 1 & 0 & -6 & 0 \end{pmatrix} N\left(0, \begin{pmatrix} 96 & 0 & 12 & 0 \\ 0 & 15 & 0 & 3 \\ 12 & 0 & 2 & 0 \\ 0 & 3 & 0 & 1 \end{pmatrix}\right) \\
&= N(0, 24)
\end{aligned}
$$

(e)
By (b), (d) and Slutsky,

$$
\sqrt{n}Y_n \xrightarrow{d} N(0, 24)
$$