**2006 Fall Part 1**

**1. Optimal critical region**
From Neyman-Pearson theorem, the optimal critical region is determined by

$$C = \left\{ x = (x_1, \cdots, x_n) \,\middle|\, \frac{\prod_{i=1}^n \theta_1 x^{\theta_1 - 1}}{\prod_{i=1}^n \theta_0 x^{\theta_0 - 1}} \geq k \right\}$$

Plugging $\theta_0 = 1$ and $\theta_1 = 2$ into the above,

$$\begin{aligned} C &= \left\{ x \,\middle|\, 2^n \prod_{i=1}^n x_i \geq k \right\} \\ &= \left\{ x \,\middle|\, \prod_{i=1}^n x_i \geq 2^{-n} k \right\} \end{aligned}$$

Letting $c = 2^{-n}k$ completes the proof.

**2. Information equality**
Differentiating $1 = \int f(x,\theta)dx$ with respect to $\theta$,

$$0 = \frac{\partial}{\partial \theta} \int f(x,\theta)dx = \int \frac{\partial f(x,\theta)}{\partial \theta} dx = \int \frac{\partial f(x,\theta)}{\partial \theta} \frac{1}{f(x,\theta)} \cdot f(x,\theta)dx = \int \left[ \frac{\partial \log f(x,\theta)}{\partial \theta} \right] f(x,\theta)dx \qquad (1)$$

Differentiating both sides with respect to $\theta$ again,

$$\begin{aligned} 0 &= \int \frac{\partial^2 \log f(x,\theta)}{\partial \theta^2} f(x,\theta)dx + \int \frac{\partial \log f(x,\theta)}{\partial \theta} \frac{\partial f(x,\theta)}{\partial \theta} dx \\ 0 &= \int \frac{\partial^2 \log f(x,\theta)}{\partial \theta^2} f(x,\theta)dx + \int \left[ \frac{\partial \log f(x,\theta)}{\partial \theta} \right]^2 f(x,\theta)dx \end{aligned}$$

where we used the same trick as in (1).

$$\begin{aligned} \int \left[ \frac{\partial}{\partial \theta} \log f(x,\theta) \right]^2 f(x,\theta)dx &= -\int \left[ \frac{\partial^2}{\partial \theta^2} \log f(x,\theta) \right] f(x,\theta)dx \\ E\left[ \frac{\partial}{\partial \theta} \log f(x,\theta) \right] &= -E\left[ \frac{\partial^2}{\partial \theta^2} \log f(x,\theta) \right] \end{aligned}$$

**3. Method of moments** From the given moment conditions, we have the following sample moment conditions

$$\overline{X} = \alpha\beta \quad , \quad \frac{1}{n} \sum_{i=1}^n (X_i - \overline{X})^2 = \alpha\beta^2$$

Therefore,

$$\begin{aligned} \widehat{\beta} &= \frac{\widehat{\alpha}\widehat{\beta}^2}{\widehat{\alpha}\widehat{\beta}} = \frac{\sum_{i=1}^n (X_i - \overline{X})^2}{n\overline{X}} \\ \widehat{\alpha} &= \frac{\overline{X}}{\widehat{\beta}} = \frac{n\overline{X}^2}{\sum_{i=1}^n (X_i - \overline{X})^2} \end{aligned}$$

**4. Summation of independent normal distriubtion**
Recall that MGF of $X_i \sim N(\mu_i, \sigma_i^2)$ is

$$M_i(t) = \exp\left( \mu_i t + \frac{1}{2}\sigma_i^2 t^2 \right)$$

MGF of $\sum_i a_i X_i$ is

$$\begin{aligned}
E\left[e^{t\sum_i a_i X_i}\right] &= E\left[e^{ta_1X_1+ta_2X_2+\cdots+ta_nX_n}\right] \\
&= E\left[e^{ta_1X_1}e^{ta_2X_2}\cdots e^{ta_nX_n}\right] \\
&= E\left[e^{ta_1X_1}\right]E\left[e^{ta_2X_2}\right]\cdots E\left[e^{ta_nX_n}\right] \qquad \because \text{independence} \\
&= \exp\left(\mu_1 ta_1 + \frac{1}{2}\sigma_1^2(ta_1)^2\right)\exp\left(\mu_2 ta_2 + \frac{1}{2}\sigma_2^2(ta_2)^2\right)\cdots\exp\left(\mu_n ta_n + \frac{1}{2}\sigma_n^2(ta_n)^2\right) \\
&= \exp\left([a_1\mu_1 + a_2\mu_2 + \cdots a_n\mu_n]t + \frac{1}{2}[a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \cdots + a_n^2\sigma_n^2]t^2\right)
\end{aligned}$$

This is MGF of normal distribution with mean $\sum_i a_i\mu_i$ and variance $\sum_i a_i^2\sigma_i^2$.

**5. Asymptotic distribution**

$E[z_i\varepsilon_i] = 0$ and thus $z_i$ and $\varepsilon_i$ are independent of each other.[1] So

$$E[z_i^2\varepsilon_i^2] = E[z_i^2]E[\varepsilon_i^2] = 1$$

Now from $E[z_i\varepsilon_i] = 0$ and $\text{var}[z_i\varepsilon_i] = 1$, applying CLT,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n z_i\varepsilon_i \xrightarrow{d} N(0,1)$$

By LLN,

$$\frac{1}{n}\sum_{i=1}^n z_i x_i \xrightarrow{p} E[z_i x_i] = \frac{1}{2}$$

By continuity[2],

$$\frac{1}{n^{-1}\sum_{i=1}^n z_i x_i} \xrightarrow{p} 2$$

By Slutsky,

$$\frac{n^{-1/2}\sum_{i=1}^n z_i\varepsilon_i}{n^{-1}\sum_{i=1}^n z_i x_i} \xrightarrow{d} N(0,4)$$

[1] This is true only when they are jointly normal.
[2] Mann-Wald theorem