2006 Spring Part 1

1. Neyman-Pearson theorem
See Hogg et. al, pp. 422-423. [1]

2. Cramer-Rao Lower Bound
See Hogg et. al, pp. 322-323.

3. Gauss-Markov theorem
Note that

$$\begin{aligned} \text{var}(\widehat{\beta}) &= \sigma^2(X'X)^{-1} \\ \text{var}(c) = E[(c-\beta)(c-\beta)'] &= E[C\varepsilon\varepsilon'C] = \sigma^2 CC' \qquad (1) \end{aligned}$$

From $E[c] = \beta$,

$$\begin{aligned} \beta = E[c] = E[Cy] = E[C(X\beta+\varepsilon)] &= CX\beta + CXE[\varepsilon] = CX\beta \\ CX &= I_k \qquad (2) \end{aligned}$$

Define $L = C - (X'X)^{-1}X'$ and $D = (X'X)^{-1}X'$. Since $C = L + D$,

$$\begin{aligned} CC' &= LL' + LD' + DL' + DD' \quad \text{where} \\ & LD' = CX(X'X)^{-1} - (X'X)^{-1}X'X(X'X)^{-1} = 0 \quad \because (2) \end{aligned}$$

Since LL' is positive semidefinite, we have

$$CC' = LL' + DD' \geq DD'$$

which, by (1), is equivalent to

$$\text{var}(c) \geq \sigma^2(X'X)^{-1}$$

4. Relation between Unifrom and the other distirubtion
Note that F is monotonely increasig function.

$$\begin{aligned} \Pr(Y \leq y) &= \Pr\left(F^{-1}(U) \leq y\right) \\ &= \Pr\left(F(F^{-1})(U) \leq F(y)\right) \\ &= \Pr\left(U \leq F(y)\right) \\ &= F(y) \end{aligned}$$

Therefore, Y has a cdf equal to F.

5. Asymptotic distribution
We can see that $X_i \sim N(0,1)$, since

$$E[e^{tX_i}] = e^{\frac{1}{2}t^2}$$

By LLN and CLT, respectively,

$$\frac{1}{n}\sum_{i=1}^{n} X_i^4 \xrightarrow{p} E[X_i^4] = 3$$

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i \xrightarrow{d} N(0,1)$$

Thus by Slutsky,

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^4\right)^2 \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \xrightarrow{d} N(0,81)$$

[1] Hogg, McKean and Craig, *Introduction to Mathematical Statistics*, 6th edition, Prentice Hall, 2005.