2006 Spring Part 2

Note that all x_i's are $1 \times k$ vector in Part 2.

1. GLS model
See Hao's solution. (week 7 note in TA link)

(a)
Since observations are independent, we can do WLS.

$$\text{Run OLS of } \frac{y_i}{\exp\left(\frac{1}{2}w_i\gamma\right)} \text{ on } \frac{x_i}{\exp\left(\frac{1}{2}w_i\gamma\right)}$$

Thus

$$\widehat{\beta}_{GLS} = \left[\sum_{i=1}^{n} \frac{x_i' x_i}{\exp(w_i\gamma)}\right]^{-1} \sum_{i=1}^{n} \frac{x_i' y_i}{\exp(w_i\gamma)}$$

(b)
Individual likelihood function is

$$f(y_i|x_i) = \frac{1}{\sqrt{2\pi\exp(w_i\gamma)}} \exp\left[-\frac{(y_i - x_i\beta)^2}{2\exp(w_i\gamma)}\right] = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}w_i\gamma - \frac{(y_i - x_i\beta)^2}{2\exp(w_i\gamma)}\right]$$

Log likelihood function is

$$\mathcal{L} = \sum_{i=1}^{n} \log f(y_i|x_i) = -\frac{n}{2}\log 2\pi - \frac{1}{2}\sum_{i=1}^{n} w_i\gamma - \frac{1}{2}\sum_{i=1}^{n} \frac{(y_i - x_i\beta)^2}{\exp(w_i\gamma)}$$

FOC is

$$\frac{\partial \mathcal{L}}{\partial \beta} = \sum_{i=1}^{n} \frac{x_i'(y_i - x_i\beta)}{\exp(w_i\gamma)} = 0 \tag{1}$$

Therefore, ML estimator is

$$\widehat{\beta}_{ML} = \left[\sum_{i=1}^{n} \frac{x_i' x_i}{\exp(w_i\gamma)}\right]^{-1} \sum_{i=1}^{n} \frac{x_i' y_i}{\exp(w_i\gamma)} \tag{2}$$

which is the same with GLS estimator obtained in (a)

(c)
Note that (i) linear model and (ii) independent observations are given in the question.

(iii) $E\left[\left\|\frac{x_i}{\exp\left(\frac{1}{2}w_i\gamma\right)}\right\|^{2+\delta}\right] < \Delta < \infty \quad \forall i$

(iv) $E\left[\left\|\frac{x_i'\varepsilon_i}{\exp(w_i\gamma)}\right\|^{1+\delta}\right] < \Delta < \infty \quad \forall i$

(v) $E(x_i'\varepsilon_i) = 0 \quad \forall i$ which implies $E\left[\frac{x_i'\varepsilon_i}{\exp(w_i\gamma)}\right] = 0 \quad \forall i$

(vi) $E\left[\frac{1}{n}\sum_{i=1}^{n} \frac{x_i' x_i}{\exp(w_i\gamma)}\right]$: uniformly p.d. which implies identification

Note here that transformed variables are not identical. Refer to lecture note 7, p.26 about the definition of uniform positive definiteness.

From (2),

$$\widehat{\beta}_{ML} - \beta = \left[\frac{1}{n}\sum_{i=1}^{n}\frac{x_i'x_i}{\exp(w_i\gamma)}\right]^{-1}\frac{1}{n}\sum_{i=1}^{n}\frac{x_i'\varepsilon_i}{\exp(w_i\gamma)} \tag{3}$$

(iii) and (vi) imply

$$\left[\frac{1}{n}\sum_{i=1}^{n}\frac{x_i'x_i}{\exp(w_i\gamma)}\right]^{-1} \xrightarrow{p} \left(\lim_{n\to\infty} E\left[\frac{1}{n}\sum_{i=1}^{n}\frac{x_i'x_i}{\exp(w_i\gamma)}\right]\right)^{-1}$$

and (iv) and (v) imply

$$\frac{1}{n}\sum_{i=1}^{n}\frac{x_i'\varepsilon_i}{\exp(w_i\gamma)} \xrightarrow{p} 0$$

For asymptotic normality, we need the following assumptions in addition to (i),(ii),(iii),(v), and (vi)

(vii) $E\left[\left\|\frac{x_i'\varepsilon_i}{\exp(w_i\gamma)}\right\|^{2+\delta}\right] < \Delta < \infty \quad \forall i$ which implies $\text{var}\left[\frac{x_i'\varepsilon_i}{\exp(w_i\gamma)}\right] < \infty \quad \forall i$

(viii) $\overline{\sigma}_n^2 \equiv E\left[\frac{1}{n}\sum_{i=1}^{n}\frac{\varepsilon_i^2 x_i'x_i}{\exp(2w_i\gamma)}\right]$: uniformly p.d.

From (3),

$$\sqrt{n}\left(\widehat{\beta}_{ML} - \beta\right) \xrightarrow{d} N\left(0, B_0^{-1}\Omega_0 B_0^{-1}\right)$$

where

$$B_0 = \lim_{n\to\infty} E\left[\frac{1}{n}\sum_{i=1}^{n}\frac{x_i'x_i}{\exp(w_i\gamma)}\right] \quad , \quad \Omega_0 = \lim_{n\to\infty}\overline{\sigma}_n^2$$

(d)

We first estimate γ. For this, run OLS of y_i on x_i and get $\widehat{\varepsilon}_{i,OLS} = y_i - x_i\widehat{\beta}_{OLS}$.
Then do NLS of $\widehat{\varepsilon}_{i,OLS}^2$ on $\exp(w_i\gamma)$ with the regression model $\widehat{\varepsilon}_{i,OLS}^2 = \exp(w_i\gamma) + \eta_i$.

$$\min \frac{1}{2}\sum_{i=1}^{n}\left[\widehat{\varepsilon}_{i,OLS}^2 - \exp(w_i\gamma)\right]^2$$

FOC is

$$-\sum_{i=1}^{n}\left[\widehat{\varepsilon}_{i,OLS}^2 - \exp(w_i\gamma)\right]\exp(w_i\gamma)w_i' = 0 \tag{4}$$

Obtain $\widehat{\gamma}_{NLS}$ and with this, do feasible GLS in the same way in (a).

(e)

Differentiating log likelihood function with respect to γ

$$\frac{\partial\mathcal{L}}{\partial\gamma} = -\frac{1}{2}\sum_{i=1}^{n}w_i' + \frac{1}{2}\sum_{i=1}^{n}(y_i - x_i\beta)^2\exp(-w_i\gamma)w_i' = 0$$

$$\sum_{i=1}^{n}\left[(y_i - x_i\beta)^2 - \exp(w_i\gamma)\right]\exp(-w_i\gamma)w_i' = 0 \tag{5}$$

$\widehat{\beta}_{ML}$ and $\widehat{\gamma}_{ML}$ are the solution to (1) and (5). Note that (5) has a similar form with (4) since $y_i - x_i\beta$ can be regarded as a residual. Of course, two equations are not the same, so $\widehat{\beta}_{ML}$ and $\widehat{\gamma}_{ML}$ will be different from $\widehat{\beta}_{NLS}$ and $\widehat{\gamma}_{NLS}$, but they would have asymptotically the same distribution. Note again that since FOC with respect to β is the same for both cases, $\widehat{\beta}_{ML}$ is obtained by (2) with $\gamma = \widehat{\gamma}_{ML}$.