1. (a)

Let D be the event that the rider uses drugs, and P be the event that the rider tests positive. Then by assumption, we have $\pi(D) = 0.7$, $\pi(P|D) = 0.9$, and $\pi(P| \sim D) = 0.2$. Thus

$$\pi(D|P) = \frac{\pi(D \cap P)}{\pi(P)} = \frac{\pi(P|D)\pi(D)}{\pi(P|D)\pi(D) + \pi(P| \sim D)\pi(\sim D)} = \frac{0.9 \times 0.7}{0.9 \times 0.7 + 0.2 \times 0.3} = 0.913$$

(b)

We have $\pi(P|D) = 1$. Let $\pi(P| \sim D) = z$, then

$$\pi(D|P) = \frac{\pi(P|D)\pi(D)}{\pi(P|D)\pi(D) + \pi(P| \sim D)\pi(\sim D)} = \frac{1 \times 0.7}{1 \times 0.7 + z \times 0.3} \geq 0.95$$

Solving the inequality, we have $z \leq 0.1228$.

(c)

We have no additional information specific to a winner, that is, there is no prior distribution that winners are more likely to use drugs nor riders who use drugs are more likely to win. Thus, the winner of Stage 14, Floyd Landis, is just one of the riders. Therefore, the probability that he was actually using drugs given he tested positive is the same as the answer to part (a), 0.913.

2.

No. They are not necessary because given priors and conditionals, there are enough information to derive $\pi(X)$ and $\pi(Y)$ using Bayes' Rule.

$$\pi(X) = \pi(X|R)\pi(R) + \pi(X|G)\pi(G) + \pi(X|B)\pi(B) = \frac{2}{3} \cdot \frac{9}{24} + \frac{4}{13} \cdot \frac{13}{24} + 1 \cdot \frac{2}{24} = \frac{1}{2}$$

$$\pi(Y) = \pi(Y|R)\pi(R) + \pi(Y|G)\pi(G) + \pi(Y|B)\pi(B) = \frac{1}{3} \cdot \frac{9}{24} + \frac{9}{13} \cdot \frac{13}{24} + 0 \cdot \frac{2}{24} = \frac{1}{2}$$

3. (a)

Intuition here is that one player with less information will never enter the bet when the other player with more information enters the bet.

Let $\omega \in B_j \subset A_i$ be the true state, and E be the event that 1 wins a and 2 loses a. If E^C occurs, 1 loses b and 2 wins b. Let also $\eta_A = P(E|A_i)$ and $\eta_B = P(E|B_j)$. Since 1 is risk averse, she will not bet if $\eta_A \cdot a + (1 - \eta_A)(-b) \leq 0$, that is $\eta_A \leq \frac{b}{a+b}$. For the same reason, 2 will not bet also if $\eta_B \cdot (-a) + (1 - \eta_B) \cdot b \leq 0$, that is $\eta_B \geq \frac{b}{a+b}$.

If $\eta_A = \eta_B$, they do not agree to bets. To see this, note that if $\eta_A = \eta_B > \frac{b}{a+b}$, 2 does not bet, and if $\eta_A = \eta_B < \frac{b}{a+b}$, 1 does not bet. In the case $\eta_A = \eta_B = \frac{b}{a+b}$, neither agent bets since both are risk averse.

Suppose $\eta_A \neq \eta_B$. Since $B_j \subset A_i$ is a common knowledge between two players, 1 reasons that there is some C such that $\omega \in C \subset A_i$ and $P(E|C) = \eta_B$. If there is only one such C, she will change her posterior as $\eta_A = P(E|C) = \eta_B$. If there are some such sets $C_1, \cdots, C_k$, she will give positive beliefs $\sigma_1, \cdots, \sigma_k$ to them. It is straightforward that $P(E|C_1) = \cdots = P(E|C_k) = \eta_B$, so $\eta_A = \sum_{t=1}^{k} \sigma_t P(E|C_t) = \eta_B$ again. On the other hand, 2 also knows that $B_j \subset A_i$, and will not change his posterior. Of course, 1 reasons that 2 will not change his posterior, and 2 reasons that 1 will change his posterior, and so on. Therefore, we have $\eta_A = \eta_B$. Neither agent agrees to bets.

(b)
Consider the game in which the row player wins \$1 if the true state is in boxes marked with W, and loses \$1 otherwise. Boxes are given by the following.

		W		
	W		W	W
W	W	*	W	W
W	W		W	
		W		

The true box is denoted by $*$. Suppose that the information sets of the row player (R) are given by each row, and that those of the column player (C) are given by each column. Suppose further that both players do not know about each other's information set.

C knows that the true box is on the 3rd column, reasons that he has $\frac{3}{5}$ chance of winning, and bets. R knows that the true box is on the 3rd row, reasons that she has $\frac{4}{5}$ chance of winning, and also bets. But players do not know each other's information partition, so they cannot update their posterior after seeing each other's behavior. Thus they solely base their betting decision on their priors and they both bet.

If we assume that both players know each other's information set, but it is not a common knowledge, it is still the case that both players do bet. In this case, C reasons that R bets since R is informed that the true box is in one of the 2nd, 3rd, and 4th rows. Thus C still bets. R also reasons that C bets since C is informed that the true box is in one of the 1st, 3rd, and 5th columns, thus R still bets. But R does not know that C knows about R's partition, so R cannot get any information from the fact that C still bets. This is the same for C, therefore, they both bet.

4. (a)
Since they know that at least one of them has a dirty face, they think that the possible states are as follows.

State	1	2	3	4	5	6	7
Ann	D			D	D		D
Bernice		D		D		D	D
Chloe			D		D	D	D

The information partitions of agents are

$$\begin{aligned} Ann &: \{1\}, \{2,4\}, \{3,5\}, \{6,7\} \\ Bernice &: \{1,2,3,4,5,6,7\} \\ Chole &: \{3\}, \{1,5\}, \{2,6\}, \{4,7\} \end{aligned}$$

If the true state is 1, Ann figures out that she has a dirty face. If it is 3, Chloe does, and Ann figures out that she does not have a dirty face. If nobody raises her hand at the 1st round, all agents reason that 1 and 3 are not the true state, so they modify their information partitions. New information partitions of agents at the 2nd round are

$$\begin{aligned} Ann &: \{2,4\}, \{5\}, \{6,7\} \\ Bernice &: \{2,4,5,6,7\} \\ Chloe &: \{5\}, \{2,6\}, \{4,7\} \end{aligned}$$

If the true state is 5, Ann figures out that her face is dirty, and so does Chloe. If nobody raises her hand at the 2nd round, all agents reason that 5 is not the true state. New information partitions at the 3rd round are

$$\begin{aligned} Ann &: \{2,4\},\{6,7\} \\ Bernice &: \{2,4,6,7\} \\ Chloe &: \{2,6\},\{4,7\} \end{aligned}$$

So Ann cannot figure out whether her own face is dirty in any state. Therefore, she can only figure out if the true state is 1,3 or 5.

(b)

If Ann and Chloe believe that Bernice can see, Ann thinks that in any states she can figure out whether her face is dirty. But if the true state is 2, then her reasoning is incorrect, and otherwise, she is correct in reasoning whether her own face is dirty.

True state	Reasoning	Is Ann correct?
1	Ann thinks that her face is dirty.	yes
2	Bernice does not raise her hand at 1st round. Ann thinks that her face is dirty at 2nd round.	no
3	Chloe raises her hand and Ann does not at 1st round.	yes
4	Ann and Bernice do not raise their hands at 1st round. Ann thinks that her face is dirty at 2nd round.	yes
5	Ann and Chloe raise their hands at 2nd round.	yes
6	With the same reasoning as in 4, C raises her hand at 2nd round. Until the game ends, Ann does not raise her hand, which is correct.	yes
7	Ann and Chloe raise their hands at 3rd round.	yes

5. (a)

Suppose that there is expected utility U consistent with given preferences. Then, we have

$$\begin{aligned} U(1A) > U(1B) &\quad\Leftrightarrow\quad U(\$1m) > 0.89U(\$1m) + 0.01U(0) + 0.1U(\$5m) \\ U(2A) < U(2B) &\quad\Leftrightarrow\quad 0.89U(0) + 0.11U(\$1m) < 0.9U(0) + 0.1U(\$5m) \end{aligned}$$

The first inequality implies $0.11U(\$1m) > 0.01U(0) + 0.1U(\$5m)$, which contradicts to $0.11U(\$1m) < 0.01U(0) + 0.1U(\$5m)$ implied by the second inequality. Therefore, these preferences are not consistent with objective expected utility.

(b)

Suppose all of three axioms hold. The axiom of *Reduction of Compound Lotteries* implies

$$1A \sim 0.89 \times \$1m + 0.11 \times \$1m$$

Thus by the axiom of *Sure Thing Principle*(independence), $1A \succ 1B$ implies

$$\$1m \succ \frac{1}{11} \times 0 + \frac{10}{11} \times \$5m,$$

which in turn implies $2A \succ 2B$. But we know that $2B \succ 2A$. Therefore, either of the axioms of *Reduction of Compound Lotteries* and *Sure Thing Principle* is violated in such preferences. In this example, however, the axiom of *Sure Thing Principle* seems more dominantly violated.

6. (a)

Suppose that there are subjective probabilities $P(R), P(G), P(B)$ and expected utility U consistent with given preferences. Without loss of generality, let $U(\$10) > U(0) = 0$. Then, we have

$$\begin{aligned} U(1A) > U(1C) \quad &\Leftrightarrow \quad P(R)U(\$10) > P(G)U(\$10) \\ U(2A) > U(2B) \quad &\Leftrightarrow \quad [P(B) + P(G)]\, U(\$10) > [P(R) + P(B)]\, U(\$10) \end{aligned}$$

The first inequality implies $P(R) > P(G)$, which contradicts to $P(G) > P(R)$ implied by the second.

(b)

Suppose all of five axioms hold. Since $1A \succ 1C$, by the axiom of *Sure Thing Principle*, we have

$$\tfrac{1}{2} \times 1A + \tfrac{1}{2} \times 1B \succ \tfrac{1}{2} \times 1C + \tfrac{1}{2} \times 1B$$

$$\tfrac{1}{2} \times [R\$10 + G0 + B0] + \tfrac{1}{2} \times [R0 + G0 + B\$10] \succ \tfrac{1}{2} \times [R0 + G\$10 + B0] + \tfrac{1}{2} \times [R0 + G0 + B\$10]$$

By the axiom of *Reversal of Order*,

$$\begin{aligned} &R \times [\tfrac{1}{2} \times \$10 + \tfrac{1}{2} \times 0] + G \times [\tfrac{1}{2} \times 0 + \tfrac{1}{2} \times 0] + B \times [\tfrac{1}{2} \times 0 + \tfrac{1}{2} \times \$10] \\ &\qquad \succ R \times [\tfrac{1}{2} \times 0 + \tfrac{1}{2} \times 0] + G \times [\tfrac{1}{2} \times \$10 + \tfrac{1}{2} \times 0] + B \times [\tfrac{1}{2} \times 0 + \tfrac{1}{2} \times \$10] \\ &R \times [\tfrac{1}{2} \times \$10 + \tfrac{1}{2} \times 0] + G \times [\tfrac{1}{2} \times 0 + \tfrac{1}{2} \times 0] + B \times [\boldsymbol{\tfrac{1}{2}} \times \$\mathbf{10} + \boldsymbol{\tfrac{1}{2}} \times \mathbf{0}] \\ &\qquad \succ R \times [\tfrac{1}{2} \times 0 + \tfrac{1}{2} \times 0] + G \times [\tfrac{1}{2} \times \$10 + \tfrac{1}{2} \times 0] + B \times [\boldsymbol{\tfrac{1}{2}} \times \$\mathbf{10} + \boldsymbol{\tfrac{1}{2}} \times \mathbf{0}] \end{aligned}$$

Again by the axiom of *Reversal of Order*,

$$\tfrac{1}{2} \times [R\$10 + G0 + B\$10] + \tfrac{1}{2} \times [R0 + G0 + B0] \succ \tfrac{1}{2} \times [R0 + G\$10 + B\$10] + \tfrac{1}{2} \times [R0 + G0 + B0]$$

By the axiom of *Sure Thing Principle*,

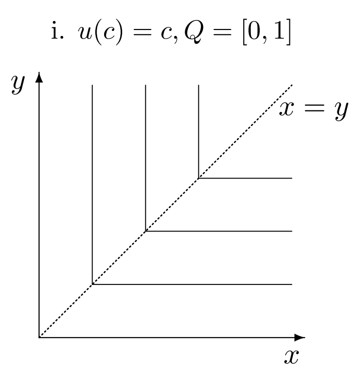
$$R\$10 + G0 + B\$10 \succ R0 + G\$10 + B\$10$$

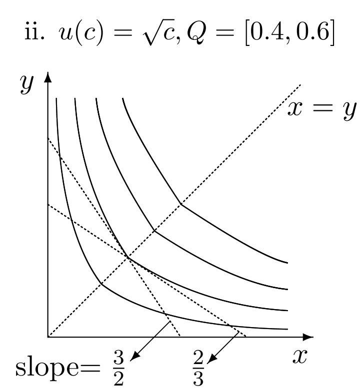
This is equivalent to $2B \succ 2A$, but it contradicts the assumption $2A \succ 2B$. Therefore, either of the axioms of *Sure Thing Principle* and *Reversal of Order* is violated in such preferences. The axiom of *Sure Thing Principle* seems to be the dominant violation.

7. (a)

$$\textbf{i.} \;\; V(x,y) = \min_{q \in [0,1]} [qx + (1-q)y] = \begin{cases} x & \text{if } x \le y \\ y & \text{if } x > y \end{cases}$$

$$\textbf{ii.} \;\; V(x,y) = \min_{q \in [0.4,0.6]} \left[q\sqrt{x} + (1-q)\sqrt{y}\right] = \begin{cases} 0.6\sqrt{x} + 0.4\sqrt{y} & \text{if } x \le y \\ 0.4\sqrt{x} + 0.6\sqrt{y} & \text{if } x > y \end{cases}$$





(b) i.
The agent always consumes $x = y$. Substituting into $x + p_y y = 1$, we have $x^* = y^* = \frac{1}{1+p_y}$. Thus the demand function is

$$d(p_y) = \left(\frac{1}{1+p_y}, \frac{1}{1+p_y} \right)$$

ii.
We have three cases.

$$d(p_y) = \begin{cases} \left(\dfrac{9p_y^2}{9p_y^2 + 4p_y}, \dfrac{4}{9p_y^2 + 4p_y} \right) & \text{if } p_y < \frac{2}{3} \\ \left(\dfrac{1}{1+p_y}, \dfrac{1}{1+p_y} \right) & \text{if } \frac{2}{3} \le p_y \le \frac{3}{2} \\ \left(\dfrac{4p_y^2}{4p_y^2 + 9p_y}, \dfrac{9}{4p_y^2 + 9p_y} \right) & \text{if } p_y > \frac{3}{2} \end{cases}$$

In dealing with contingent commodities, by expected utility properties

$$\frac{\pi_1 u'(c_1)}{\pi_2 u'(c_2)} = \frac{p_1}{p_2}$$

if objective probabilities π_1 and π_2 exist. In such cases, if the price of one commodity rises, the demand for this commodity decreases and the demand for the other commodity usually increases. But in these examples, even if the price of one commodity rises, the demand for the other could also decrease. In case (i), the demand for x always decreases as p_y increases. In case (ii), the demand for x decreases as p_y increases when $p_y \in (\frac{2}{3}, \frac{3}{2})$.

8. (a)
For the row player, D is strictly dominated by both U and M.
For the column player, R is strictly dominated by both L and M.
But there is no strictly dominant strategy for any player.

(b)
The row player never plays D, and the column player never plays R. Then, we have the modified game as follows.

	L	M
U	4, 2	2, 3
M	2, 1	1, 2

Here, in the reduced form of the game, M and L are strictly dominated by U and M respectively; the row player never plays M, and the column player never plays L. (U,M) is the Nash equilibrium obtained by iterative elimination of strictly dominated strategies.

9. (a)
There is no pure strategy dominance relation for any player. However, there exists a pure strategy strictly dominated by mixing the other pure strategies. For the row player, playing 0.6 U and 0.4 M yields $(4.2, 3.8, 1.4)$, strictly dominating D. For the column player, playing $\frac{4}{7}$ L and $\frac{3}{7}$ R yields $(\frac{29}{7}, \frac{76}{7}, 20)$, strictly dominating M.

(b)

	L	M	R
U	5, $\boxed{5}$	1, 4	1, 3
M	3, 10	8, 8	2, $\boxed{12}$
D	4, 2	2, 19	0, $\boxed{44}$

Circles are the row player's best responses for each of the column player's pure strategy, and boxes are the column player's best responses. Therefore, the Nash equilibria are (U,L) and (M,R).

10. (a)
Let q_1, q_2, q_3 be the output for firms 1,2,3, respectively. Firm 1's output is the solution of

$$\max_{q_1}[q_1(1 - q_1 - q_2 - q_3)]$$

From FOC, we have

$$q_1 = \frac{1 - q_2 - q_3}{2}$$

or

$$2q_1 + q_2 + q_3 = 1$$

Similarly, $q_1 + 2q_2 + q_3 = 1$ and $q_1 + q_2 + 2q_3 = 1$ for firms 2 and 3, respectively. Solving equations yields

$$p = q_1 = q_2 = q_3 = \frac{1}{4}$$

which is less than in 2 firms case and the total industry profit is $\frac{3}{16}$, which is also less than in 2 firms case.

(b)
In similar way, firm i's FOC is

$$q_i + \sum_{j=1}^{n} q_j = 1$$

Solving equations, we have for all i,

$$p = q_i = \frac{1}{n+1}$$

and the total industry profit is $\frac{n}{(n+1)^2}$. Both the market price and the total industry profit converge to 0 as n goes to ∞. As the number of firms increases, each firm becomes price-taker, which means that the market becomes perfectly competitive. Perfect competition drives the price to equal the marginal cost which is 0 and so the profit would also be 0.

11. (a)

Let x be the output and define the profit function $\pi(x) = xF(x+q)$. Note that $F'(x+q) < 0$, $x > 0$, and $F''(x+q) \leq 0$.

$$\pi'(x) = F(x+q) + xF'(x+q)$$
$$\pi''(x) = 2F'(x+q) + xF''(x+q) < 0$$

Since $\pi'(x)$ is strictly decreasing, x^* such that $\pi'(x^*) = 0$ is unique if it exists. Note that $0 \leq q \leq 1$ and $F'(Q) < 0$.

$$\pi'(0) = F(q) \geq 0$$
$$\pi'(1-q) = F(1) + (1-q)F'(1) \leq 0$$

Since $\pi'(x)$ is continuous, by Intermediate Value Theorem, there exists $x^* \in [0, 1-q]$ such that $\pi'(x^*) = 0$. Therefore, a best response $\beta(q) = x^*$ is unique.

(b)

$\pi(x, q)$ is continuous in both x and q and $B(q) = \{x | 0 \leq x \leq 1 - q\}$ is continuous in q and compact-valued. $\beta(q) = x^*$ is the single(unique) solution of the following maximization problem.

$$\max_{x \in B(q)} [\pi(x, q)]$$

Therefore, by Berge's Maximum Theorem, β is continuous in q, and $\beta \circ \beta$ is also continuous in q.

(c)

Define $f(q) = \beta \circ \beta(q) - q$. Since $F(1) = 0$, we have $0 < \beta(q) < 1$ for all $q \in [0, 1)$ and $\beta(1) = 0$.

$$f(0) = \beta[\beta(0)] > 0$$
$$f(1) = \beta[\beta(1)] - 1 = \beta(0) - 1 < 0$$

Since $f(q)$ is continuous, by Intermediate Value Theorem, there exists $q^* \in (0, 1)$ such that $f(q^*) = 0$, that is $\beta \circ \beta(q^*) = q^*$.

It immediately follows that $(q^*, \beta(q^*))$ is a Nash equilibrium. To prove that (q^*, q^*) is a Nash equilibrium, we need to show that $\beta(q^*) = q^*$. Differentiating the identity $\pi'[\beta(q)] = 0$ with respect to q, we have

$$\beta'(q) = -\frac{F'[\beta(q) + q] + \beta(q)F''[\beta(q) + q]}{2F'[\beta(q) + q] + \beta(q)F''[\beta(q) + q]} \in (-1, 0)$$

since $F' < 0$, $F'' \leq 0$ and $\beta > 0$. Thus, for any $q_1 > q_0$, $q_1 - q_0 > \beta(q_0) - \beta(q_1) > \beta[\beta(q_1)] - \beta[\beta(q_0)]$, or $\beta \circ \beta(q_0) - q_0 > \beta \circ \beta(q_1) - q_1$. Since $f(q) = \beta \circ \beta(q) - q$ is strictly decreasing in q, q^* such that $\beta \circ \beta(q^*) = q^*$ is unique. In similar way, there exists a unique $\widehat{q} \in (0, 1)$ such that $\beta(\widehat{q}) = \widehat{q}$. Since $\beta \circ \beta(\widehat{q}) = \beta(\widehat{q}) = \widehat{q}$ and q^* is unique, it follows that $\widehat{q} = q^*$, thus $\beta(q^*) = q^*$. Therefore (q^*, q^*) is the Nash equilibrium.

If F is not concave, the concavity of $\pi(q)$ cannot be derived, so the uniqueness of $\beta(q)$ is not guaranteed. In this case, $\beta(q)$ may not be continuous in q, and therefore, a Nash equilibrium may not exist.