MICRO HW #2

1.
ROW plays $xC+(1-x)D$ and COL plays $yC+(1-x)D$. ROW wants COL to be indifferent between C and D. Since COL's payoff is x at C, and $2(1-x)$ at D,

$$x = 2(1-x)$$

$$x = \frac{2}{3}$$

Also, COL wants ROW to be indifferent between C and D. Since ROW's payoff is $2y$ at C, and $1-y$ at D,

$$2y = 1-y$$

$$y = \frac{1}{3}$$

So the unique Nash equilibrium is that ROW plays C with probability $\frac{2}{3}$ and D with probability $\frac{1}{3}$, and COL plays C with probability $\frac{1}{3}$ and D with probability $\frac{2}{3}$.

2. (a)

	L		M		R	
U	(2)	1	0	0	(1)	[1.5]
D	0	0	(1)	[2]	0	-1.5

Each agent's best response to each other's pure strategy is depicted in the above table. (U,R) and (D,M) are Nash equilibria in pure strategies.

(b)
Case (i). COL randomizes between L and M with y and $1-y$, respectively.
ROW will be indifferent between U and D if

$$2y = 1-y$$

$$y = \frac{1}{3}$$

ROW also randomizes to make COL indifferent between L and M, thus

$$x = 2(1-x)$$

$$x = \frac{2}{3}$$

We need to check $\frac{1}{3}L+\frac{2}{3}M$ is really a best response of COL to $\frac{2}{3}C+\frac{1}{3}D$ of ROW. If COL uses R, then COL's expected payoff is $\frac{2}{3}\times 1.5+\frac{1}{3}\times(-1.5)=0.5$, which is less than $\frac{2}{3}$, the payoff when COL plays $\frac{1}{3}L+\frac{2}{3}M$. Thus $(\frac{2}{3}U+\frac{1}{3}D, \frac{1}{3}L+\frac{2}{3}M)$ is a Nash equilibrium.

Case (ii). COL randomizes between L and R.
In this case, ROW will always choose U, since it is a strictly dominant strategy. In response, COL will choose R, which is not a totally mixed strategy. There is no Nash equilibrium in which COL randomizes between L and R.

Case (iii). COL randomizes between M and R.
From ROW's indifference condition,

$$1 - y = y$$
$$y = 0.5$$

Also from COL's indifference condition,

$$2(1 - x) = 1.5x - 1.5(1 - x)$$
$$x = 0.7$$

But when ROW plays $0.7U + 0.3D$, COL's best response is L, since playing L gives COL a payoff of 0.7 while playing $0.5M + 0.5R$ gives only 0.6.

Therefore, the only Nash equilibrium in which COL randomizes over two strategies is $(\frac{2}{3}U + \frac{1}{3}D, \frac{1}{3}L + \frac{2}{3}M)$.

(c)
In order for COL to randomize over three strategies, COL should be indifferent among L, M and R.

$$x = 2(1 - x) = 1.5x - 1.5(1 - x)$$

There is no x satisfying above conditions. Therefore, there is no Nash equilibrium in which COL randomizes over three strategies.

3.
First of all, find Nash equilibria in pure stratgies.

X

	L	R
U	4 4 4	2 0 0
D	0 2 0	0 0 0

Y

	L	R
U	1 1 1	0 2 0
D	2 0 0	1 1 1

Therefore, (U, L, X) and (D, R, Y) are Nash equilibria in pure strategies.

Next, find Nash equilibria in which all players randomizes. ROW plays $xU + (1 - x)D$, COL plays $yL + (1 - y)R$ and MAT plays $zX + (1 - z)Y$. Everybody should be indifferent between their own pure strategies.

$$\begin{aligned} \text{ROW}: &\quad 4yz + 2(1-y)z + y(1-z) = 2y(1-z) + (1-y)(1-z) \\ \text{COL}: &\quad 4xz + 2(1-x)z + x(1-z) = 2x(1-z) + (1-x)(1-z) \\ \text{MAT}: &\quad 4xy = xy + (1-x)(1-y) \end{aligned}$$

Solving equations yields $z = 2 \pm \sqrt{3}$, so $z = 2 - \sqrt{3}$ since $0 < z < 1$. Also, $x = y = \frac{-1+\sqrt{3}}{2}$. Therefore,

$$\left(\frac{-1+\sqrt{3}}{2}U + \frac{3-\sqrt{3}}{2}D \ , \ \frac{-1+\sqrt{3}}{2}L + \frac{3-\sqrt{3}}{2}R \ , \ (2-\sqrt{3})X + (-1+\sqrt{3})Y \right)$$

is a Nash equilibrium.

We only need to check if there is another Nash equilibrium. Consider the case in which only one player randomizes and the other two use pure strategy. If only MAT randomizes, MAT should be

indifferent between X and Y in response to ROW's and COL's strategies. $(U, R, zX + (1-z)Y)$ and $(D, L, zX + (1-z)Y)$ are possible cases. But when ROW plays U and MAT randomizes, COL will choose L, since $4 > 2$. Also when COL plays L and MAT randomizes, ROW will choose U, since $4 > 2$. Both cannot be Nash equilibrium. If only ROW randomizes, there should be a pair of pure strategies of COL and MAT under which ROW is indifferent between U and D. But there is no such pair of strategies. The same logic applies to COL.
Also consider the case in which two players randomizes and the other uses pure strategy. If MAT plays X, both ROW and COL have strictly dominant strategies U and L in the reduced game X, thus nobody randomizes. The same logic applies when MAT plays Y. If COL plays L, MAT will never plays Y as long as ROW randomizes since Y is weakly dominated by X in the reduced game L. The same logic applies when COL plays R. IF ROW plays pure strategy, MAT will never randomize as long as COL randomizes for the same reason. Therefore, there is no Nash equilibria when one or two players randomize.

COL

ROW		L	R
	U	4 4	2 0
	D	0 2	0 0

reduced game X

MAT

ROW		X	Y
	U	4 4	1 1
	D	0 0	2 0

reduced game L

4. (a)
Consider the case in which a fraction p of Cop(C) patrols. Note that this is not a probability. If Robber(R) Burgles(B), R will be caught with probability $f(p)$, in which case R's payoff is -1 and C's payoff is 5. If R plays B and is not caught whose probability is $1 - f(p)$, R's payoff is 5 and C's payoff is 0. Thus R's expected payoff is $-f(p) + 5[1 - f(p)]$, and C's expected payoff is $5f(p)$. If R stays Home(H), R will never be caught, so R's payoff is always 0. When R plays H, a fraction p of C receives a payoff -1 due to no gain from patrol, while a fraction $1 - p$ of C receives 1 due to no crime and no patrol. Thus C's average per capita payoff is $-p + 1 - p$. The payoff matrix is

Cop

Robber		p
	B	$5 - 6f(p)$, $5f(p)$
	H	0 , $1 - 2p$

where the left is R's payoff and the right is C's payoff.

(b)
If $f(p) = p$, this game is equivalent to the game we analyzed in the class. The only change is that C's pure strategy in this game is C's mixed strategy in the original game. There is no pure strategy equilibrium in this game. Let $rB + (1-r)H$ be R's mixed strategy, and find Nash equilibria in which C plays pure strategy p and R plays a mixed strategy. For R to use a mixed strategy, R should be indifferent between B and H,

$$5 - 6f(p) = 0$$

Since $f(p) = p$, we have $p = \frac{5}{6}$. This p should maximize C's payoff

$$\max_p \{r \cdot 5f(p) + (1-r)(1-2p)\}$$

For this to be maximized at $p = \frac{5}{6}$, this should always be constant since this is linear in p. Thus $r = \frac{2}{7}$, which is the same we obtained in the class. The Nash equilibirum is $\left(\frac{2}{7}B + \frac{5}{7}H, p = \frac{5}{6}\right)$.

(c)
C's objective function is

$$\max_p \left\{5rp^2 + (1-r)(1-2p)\right\}$$

This is convex function in p, thus p should be either 0 or 1. However, if $p=0$, R's best response is B, thus C will choose $p=1$ in this response. Also in the case $p=1$, R's best response is H, so C will never choose $p=1$. Therefore, there is no Nash equilibrium in which C plays pure strategy.

(d)
R should be indifferent between B and H,

$$5 - 6\sqrt{p} = 0$$

which yields $p = \left(\frac{5}{6}\right)^2$. C's objective function is

$$\max_p \{5r\sqrt{p} + (1-r)(1-2p)\}$$

FOC is

$$\frac{5r}{2\sqrt{p}} - 2(1-r) = 0$$

This condition should be satisfied at $p = \left(\frac{5}{6}\right)^2$. Thus $r = \frac{2}{5}$. The Nash equilibrium is $\left(\frac{2}{5}B + \frac{3}{5}H, p = \frac{25}{36}\right)$.

Additional Question
Next, consider the case the penalty for being caught changes to -100.
In (b), there is no effect on r, as seen in the class. Only p changes to $p = \frac{1}{21}$ in order to make R indifferent between B and H. The new Nash equilibrium is $\left(\frac{2}{7}B + \frac{5}{7}H, p = \frac{1}{21}\right)$. As expected, robbers' behavior do not change while the fraction of cops who go on patrol decreases.
In (c), there is no Nash equilibrium, either.
In (d), p sharply decreases to $p = \left(\frac{1}{21}\right)^2$, which in turn effects r.

$$r = \frac{4\sqrt{p}}{5 + 4\sqrt{p}} = \frac{4}{109}$$

We can see that increasing penalty for robbers greatly reduces the crime rate in this case. The new Nash equilibrium is $\left(\frac{4}{109}B + \frac{105}{109}H, p = \frac{1}{441}\right)$. Generally, if the penalty is $-x$, then $p = \left(\frac{5}{5+x}\right)^2$, and $r = \frac{4}{9+x}$, thus, it can be shown that r is strictly decreasing in x.

5. (a)
Players $N = \{S, B\}$
Types $T_S = \{H, L\}$ and $T_B = \{B\}$
Priors $\mu(H) = 0.3, \mu(L) = 0.7$
Action Sets $A_S = \{p = 5, p = 2\}, A_B = \{A, R\}$
Utility u over Action Sets and Types

$$\begin{array}{ll}
u_S(5, A; H) = 1 & u_B(5, A; H) = 1 \\
u_S(5, A; L) = 4 & u_B(5, A; L) = -2 \\
u_S(2, A; H) = -2 & u_B(2, A; H) = 4 \\
u_S(2, A; L) = 1 & u_B(2, A; L) = 1 \\
u_S(\cdot, R;\ \cdot\ ) = 0 & u_B(\cdot, R;\ \cdot\ ) = 0
\end{array}$$

Strategies of behavioral game are defined as
$B_S = \{(5|H, 5|L), (5|H, 2|L), (2|H, 5|L), (2|H, 2|L)\}$ and $B_B = \{(A5, A2), (A5, R2), (R5, A2), (R5, R2)\}$.

(b)

The payoff matrix of the extended game is

Buyer

Seller	$A5\ A2$		$A5\ R2$		$R5\ A2$		$R5\ R2$	
$5\|H\ 5\|L$	3.1	−1.1	3.1	−1.1	0	0	0	0
$5\|H\ 2\|L$	1	1	0.3	0.3	0.7	0.7	0	0
$2\|H\ 5\|L$	2.2	−0.2	2.8	−1.4	−0.6	1.2	0	0
$2\|H\ 2\|L$	0.1	1.9	0	0	0.1	1.9	0	0

The Nash equilibrium in pure strategies is ($(5|H, 5|L)$, $(R5, R2)$). In this equilibrium, the expected payoffs to each player is 0 for every type. This is not an interesting equilibrium.
Find the Nash equilibrium in mixed strategies. From here on, we will shorten each pure strategy as actions in order of getting $p = 5$ and $p = 2$ for B and H and L for S. In other words, we will denote S's strategy $(5|H, 5|L)$ as 55. We can see that AR is strictly dominated by RA, and RR is weakly dominated by RA. In particular, B is indifferent between RR and RA only when S plays 55. So if S randomizes over any two of 4 strategies, B will never play RR. Thus B randomizes over AA and RA. Given that condition, 25 is strictly dominated by 55 and 22 is also strictly dominated by 52. Thus the reduced game obtained from eliminating dominated strategies is

Buyer

Seller	$A5\ A2$		$R5\ A2$	
$5\|H\ 5\|L$	3.1	−1.1	0	0
$5\|H\ 2\|L$	1	1	0.7	0.7

S plays $x55 + (1 - x)52$ and B plays $yAA + (1 - y)RA$. In order to make both players indifferent between two pure strategies, following conditions must hold.

$$\begin{aligned} -1.1x + 1 \cdot (1 - x) &= 0.7(1 - x) \\ 3.1y &= 1 \cdot y + 0.7(1 - y) \end{aligned}$$

$$x = \frac{3}{14} \quad \text{and} \quad y = \frac{1}{4}$$

Thus the Bayesian Nash equilibrium in mixed strategies is

$$\left(\frac{3}{14}(5|H, 5|L) + \frac{11}{14}(5|H, 2|L) \quad , \quad \frac{1}{4}(A5, A2) + \frac{3}{4}(R5, A2) \right)$$

In the form of mixed behavioral strategies, the equilibrium can be written as

$$\left(5 \,\middle|\, H \;\&\; \left[\frac{3}{14}(p = 5) + \frac{11}{14}(p = 2) \right] \middle|\, L \quad , \quad \frac{1}{4}(A5, A2) + \frac{3}{4}(R5, A2) \right)$$

Therefore, in this equilibrium, the utility of each type of player is

$$\begin{aligned} u_H &= \frac{1}{4} \times 1 + \frac{3}{4} \times 0 = \frac{1}{4} \\ u_L &= \frac{3}{14} \times \left(\frac{1}{4} \times 4 + \frac{3}{4} \times 0 \right) + \frac{11}{14} \times 1 = 1 \end{aligned}$$

$$u_B = 0.3 \times \frac{1}{4} \times 1 + 0.7 \times \left\{ \frac{3}{14} \times \frac{1}{4} \times (-2) + \frac{11}{14} \times 1 \right\} = \frac{11}{20}$$

6. (a)

Players $N = \{S, B\}$

Types $T_S = \{H, L\}$ and $T_B = \{D, N\}$: detector, non-detector

Priors $\mu(H, D) = 0.12, \mu(H, N) = 0.18, \mu(L, D) = 0.28, \mu(L, N) = 0.42$

Action Sets $A_S = \{p = 5, p = 2\}, A_B = \{A, R\}$

Utility u over Action Sets and Types

$$\begin{array}{ll} u_S(5, A; H, \cdot) = 1 & u_B(5, A; H, \cdot) = 1 \\ u_S(5, A; L, \cdot) = 4 & u_B(5, A; L, \cdot) = -2 \\ u_S(2, A; H, \cdot) = -2 & u_B(2, A; H, \cdot) = 4 \\ u_S(2, A; L, \cdot) = 1 & u_B(2, A; L, \cdot) = 1 \\ u_S(\cdot, R; \ \cdot \ ) = 0 & u_B(\cdot, R; \ \cdot \ ) = 0 \end{array}$$

Strategies of sellers of behavioral game are defined as

$$B_S = \{(5|H, 5|L), (5|H, 2|L), (2|H, 5|L), (2|H, 2|L)\}$$

Buyers have 64 strategies since there are 2 types of buyers and D type of buyer can assign two actions to each information in $\{(5|H), (5|L), (2|H), (2|L)\}$, and N type of buyer can assign two actions to each information in $\{5, 2\}$. The set of strategies is

$$B_B = \{(A5H, A5L, A2H, A2L|D : A5, A2|N), \cdots, (R5H, R5L, R2H, R2L|D : R5, R2|N)\}$$

However, assigning actions to $2|H$ is irrelevant in the sense that in the equilibrium, S will never play $2|H$. So we will write B's strategies in the form of $A5H, R5L, A2$ without distinguishing the actions of B when he observes $2|H$ or $2|L$.

$$B_B = \{(A5H, A5L, A2|D : A5, A2|N), \cdots, (R5H, R5L, R2|D : R5, R2|N)\}$$

(b)

There is 1 Nash equilibrium in pure strategies.

$$\Big((5|H, 5|L) \ , \ (A5H, R5L, R2|D : R5, R2|N) \Big) : u_H = 0.4, u_L = 0, u_D = 0.3, u_N = 0$$

Since $R2|D$ and $R2|N$ are out-of-equilibrium-path, B cannot get better off by changing these. But these out-of-equilibrium-path beliefs play a crucial role to support the above as a Nash equilibrium. Of course, this equilibrium is stupid in the sense that B's strategy is weakly dominated by another strategy, for example, $(A5H, R5L, A2|D : R5, A2|N)$.

Find Nash equilibrium in mixed strategies. Note that if S randomizes over any two of pure strategies, B will never play weakly dominated strategies in this game. Especially, D type of B will use $(A5H, R5L, A2)$ only, and N type of B will use $(A5, A2)$ or $(R5, A2)$. Since D uses only one strategy, we denote B's strategies in terms of strategies of N, that is, AA and RA. If B plays AA and RA only, S will never play $(2|H, 5|L)$ and $(2|H, 2|L)$. Also denote the weakly dominant strategies of S as 55 and 52. Then, we have payoff matrix as

		Buyer			
		$A5\ A2$		$R5\ A2$	
Seller	$5\|H\ 5\|L$	1.98	−0.54	0.12	0.12
	$5\|H\ 2\|L$	1	1	0.82	0.82

S plays $x55+(1-x)52$ and B plays $yAA+(1-y)RA$. In order to make both players indifferent between two pure strategies, following conditions must be satisfied.

$$\begin{aligned} -0.54x + 1\cdot(1-x) &= 0.12x + 0.82(1-x) \\ 1.98y + 0.12(1-y) &= 1\cdot y + 0.82(1-y) \end{aligned}$$

$$x = \frac{3}{14} \quad \text{and} \quad y = \frac{5}{12}$$

Thus the Bayesian Nash equilibrium in mixed strategy is

$$\left(\frac{3}{14}(5|H,5|L) + \frac{11}{14}(5|H,2|L)\ ,\ \frac{5}{12}(A5H, R5L, A2|D : A5, A2|N) + \frac{5}{12}(A5H, R5L, A2|D : R5, A2|N)\right)$$

In the form of mixed behavioral strategies, the equilibrium can be written as

$$\left(5\,\middle|H\ \&\ \left[\frac{3}{14}(p=5) + \frac{11}{14}(p=2)\right]\middle|\ L\quad ,\quad (A5H, R5L, A2)\,\middle|D\ \&\ \left[\frac{5}{12}(A5, A2) + \frac{7}{12}(R5, A2)\right]\middle|\ N\right)$$

Therefore, in this equilibrium, the utility of each type of player is

$$\begin{aligned} u_H &= 0.4\times 1 + 0.6\times\left(\frac{5}{12}\times 1 + \frac{7}{12}\times 0\right) = \frac{13}{20} \\ u_L &= \frac{3}{14}\times\left\{0.4\times 0 + 0.6\times\left(\frac{5}{12}\times 4 + \frac{7}{12}\times 0\right)\right\} + \frac{11}{14}\times 1 = 1 \\ u_D &= 0.3\times 1 + 0.7\times\left(\frac{3}{14}\times 0 + \frac{11}{14}\times 1\right) = \frac{17}{20} \\ u_N &= 0.3\times\frac{5}{12}\times 1 + 0.7\times\left\{\frac{3}{14}\times\frac{5}{12}\times(-2) + \frac{11}{14}\times 1\right\} = \frac{11}{20} \end{aligned}$$

(c)
The only change is that the probability that N type of B rejects to buy a car of $p=5$ decreases. (they accept cars at $p=5$ with higher probability). The presence of D type of B penalyzes cheating S by not buying a car of low quality at a high price. So in order to make S indifferent between cheating and being honest, N should increase the portion of buying a low quality car at a high price. This increases the transaction, which increases social efficiency, since every car is valued higher to buyers than to sellers.

7. (a)
Players $N = \{A, B\}$
Types $T_A = \{R, B\}$ and $T_B = \{\cdot\}$
Priors $\mu(R) = 0.5, \mu(B) = 0.5$
Action Sets $A_A = \{P, B\}, A_B = \{F, C\}$

Utility u over Action Sets and Types

$$\begin{array}{ll} u_A(P,\cdot;R)=1 & u_B(P,\cdot;R)=-1 \\ u_A(P,\cdot;B)=-1 & u_B(P,\cdot;R)=1 \\ u_A(B,F;\cdot)=1 & u_B(B,F;\cdot)=-1 \\ u_A(B,C;R)=2 & u_B(B,C;R)=-2 \\ u_A(B,C;B)=-2 & u_B(B,C;B)=2 \end{array}$$

Strategies of behavioral game are defined as

$$B_A=\{(P|R,P|B),(P|R,B|B),(B|R,P|B),(B|R,B|B)\} \text{ and } B_B=\{F,C\}$$

(b)

The payoff matrix of the extended game is

A \ B	F		C	
$P\|R\ P\|B$	0	0	0	0
$P\|R\ B\|B$	1	-1	$-\frac{1}{2}$	$\frac{1}{2}$
$B\|R\ P\|B$	0	0	$\frac{1}{2}$	$-\frac{1}{2}$
$B\|R\ B\|B$	1	-1	0	0

There is no Nash equilibrium in pure strategies in this game. There is also no Nash equilibrium at which one randomizes and the other plays pure. To find Nash equilibria in totally mixed strategies, we can eliminate weakly dominated strategies. A will never use PP and PB as long as B randomizes over F and C since PP is weakly dominated by BP, and PB by BB. A plays $xBP+(1-x)BB$ and B plays $yC+(1-y)F$. To make both players indifferent between two pure strategies they have,

$$\begin{aligned} -(1-x) &= -\frac{1}{2}x \\ \frac{1}{2}(1-y) &= y \end{aligned}$$

$$x=\frac{2}{3} \quad \text{and} \quad y=\frac{1}{3}$$

The Bayesian Nash equilibrium is

$$\left(\frac{2}{3}(B|R,P|B)+\frac{1}{3}(B|R,B|B)\ ,\ \frac{1}{3}F+\frac{2}{3}C\right)$$

8. (a)

We now assume that the mechanism has to satisfy anonymity and balancedness. The mechanism consists of the probability of road construction p and the cost allocation a. To be socially efficient, p should depend on n and c only, where n is the number of H type firms.

$$p(n,c)=\begin{cases} 1 & \text{if } n>c \\ 0 & \text{otherwise} \end{cases}$$

To be anonymous, a should depend on the firm's report r and the number of reported H type firms n. Thus $a(r,n)$ is

$$\begin{aligned}
a(H;1) &= c \cdot p(1,c) \\
a(H;2) &= \frac{c}{2} \cdot p(2,c) \\
a(H;3) &= \frac{c}{3} \cdot p(3,c) \\
a(L;\cdot) &= 0
\end{aligned}$$

where the last condition guarantees individual rationality. This definition of c means that every firm pays only when the road is indeed constructed. We have three cases.

(i) $0 < c < 1$
The road should be constructed if $n \geq 1$. Since valuation and prior is symmetric and a is anonymous, we only need to consider one firm f's incentive. If f is H, f's expected payoff when it reports H should be no less than when it reports L untruthfully.

$$\begin{aligned}
&Pr(n=3|f=H)[1-a(H;3)] + Pr(n=2|f=H)[1-a(H;2)] + Pr(n=1|f=H)[1-a(H;1)] \\
&\qquad \geq Pr(n=3|f=H)\cdot 1 + Pr(n=2|f=H)\cdot 1 + Pr(n=1|f=H)\cdot 0
\end{aligned}$$

$$\begin{aligned}
h^2\left(1-\frac{c}{3}\right) + 2h(1-h)\left(1-\frac{c}{2}\right) + (1-h)^2(1-c) &\geq h^2\cdot 1 + 2h(1-h)\cdot 1 + (1-h)^2\cdot 0 \\
h^2\frac{c}{3} + 2h(1-h)\frac{c}{2} + (1-h)^2 c &\leq (1-h)^2 \\
c &\leq \frac{3(1-h)^2}{3-3h+h^2}
\end{aligned}$$

We can easily check that $0 < \frac{3(1-h)^2}{3-3h+h^2} < 1$. So if $\frac{3(1-h)^2}{3-3h+h^2} < c < 1$, the firm wants to lie rather than reports truthfully. Of course, we need to check whether f will falsely report H, when it is L in fact.

$$0 \geq Pr(n=3|f=H)[-a(H;3)] + Pr(n=2|f=H)[-a(H;2)] + Pr(n=1|f=H)[-a(H;1)]$$

But this is trivial since $a(H;\cdot) \geq \frac{c}{3} > 0$.

(ii) $1 \leq c < 2$
The road should be constructed if $n \geq 2$. The incentive compatibility condition is

$$\begin{aligned}
h^2\left(1-\frac{c}{3}\right) + 2h(1-h)\left(1-\frac{c}{2}\right) + (1-h)^2\cdot 0 &\geq h^2\cdot 1 + 2h(1-h)\cdot 0 + (1-h)^2\cdot 0 \\
h^2\frac{c}{3} + 2h(1-h)\frac{c}{2} &\leq 2h(1-h) \\
c &\leq \frac{6-6h}{3-2h}
\end{aligned}$$

It is obvious that $0 < \frac{6-6h}{3-2h} < 2$. If $\frac{6-6h}{3-2h} < c < 2$, the firm doesn't want to report truthfully.

(iii) $2 \leq c < 3$
The road should be constructed if $n = 3$. The incentive compatibility condition is

$$\begin{aligned}
h^2\left(1-\frac{c}{3}\right) + 2h(1-h)\cdot 0 + (1-h)^2\cdot 0 &\geq h^2\cdot 0 + 2h(1-h)\cdot 0 + (1-h)^2\cdot 0 \\
h^2\left(1-\frac{c}{3}\right) &\geq 0
\end{aligned}$$

This condition always holds. Therefore in this case, the firm will report its true type.

In summary, there exists such mechanism only if

$$0 < c \leq \frac{3(1-h)^2}{3-3h+h^2} \quad \text{or} \quad 1 \leq c \leq \frac{6-6h}{3-2h} \quad \text{or} \quad 2 \leq c < 3$$

where c satisfying the second condition may not exist if $h > \frac{3}{4}$. We can see that as h gets higher, the range of c gets smaller, which seems reasonable. The firm can be better off by misreporting if the probability that other firms are H increases, since the probability that the road will be constructed does not decrease so much, but misreporting saves its cost.

(b)
For such c obtained in (a), we can use $p(n,c)$ and $a(r,n)$ defined in (a). Under this mechanism, the truth report is one of Nash equilibria if c falls in the range obtained in (a).

(c)
It does not matter. If we allow non-anonymous mechanism, we should consider a_1, a_2, a_3 as cost allocation function. The firm reporting L should pay 0. Consider the case $0 < c < 1$. The incentive compatibility conditions are

$$\begin{aligned}
h^2[1-a_1(H,H,H)] + h(1-h)[1-a_1(H,H,L)] + h(1-h)[1-a_1(H,L,H)] \\
+(1-h)^2[1-a_1(H,L,L)] \quad &\geq \quad h^2 \cdot 1 + 2h(1-h)^2 \cdot 1 + (1-h)^2 \cdot 0 \\
h^2[1-a_2(H,H,H)] + h(1-h)[1-a_2(H,H,L)] + h(1-h)[1-a_2(L,H,H)] \\
+(1-h)^2[1-a_2(L,H,L)] \quad &\geq \quad h^2 \cdot 1 + 2h(1-h)^2 \cdot 1 + (1-h)^2 \cdot 0 \\
h^2[1-a_3(H,H,H)] + h(1-h)[1-a_3(H,L,H)] + h(1-h)[1-a_3(L,H,H)] \\
+(1-h)^2[1-a_3(L,L,H)] \quad &\geq \quad h^2 \cdot 1 + 2h(1-h)^2 \cdot 1 + (1-h)^2 \cdot 0
\end{aligned}$$

Firm 2 and firm 3 also have similar conditions. We also have budget balancedenss condition.

$$\begin{aligned}
a_1(H,H,H) + a_2(H,H,H) + a_3(H,H,H) = c \\
a_1(H,H,L) + a_2(H,H,L) = c \\
a_1(H,L,H) + a_3(H,L,H) = c \\
a_2(L,H,H) + a_3(L,H,H) = c \\
a_1(H,L,L) = a_2(L,H,L) = a_3(L,L,H) = c
\end{aligned}$$

All of the above 14 conditions together imply

$$h^2(3-c) + h(1-h)(6-3c) + (1-h)^2(3-3c) \geq 3h^2 + 6h(1-h)$$

which is equivalent to

$$c \leq \frac{3(1-h)^2}{3-3h+h^2}$$

This is the same condition we obtained in (a) case (i). If $\frac{3(1-h)^2}{3-3h+h^2} < c < 1$, there does not exist non-anonymous mechanism that is socially efficient, individually rational and incentive compatible. This means that every anonymous mechanism can achieve the outcome which is achievable with non-anonymous mechanism. We can also obtain the same condition for the case $1 \leq c < 2$ as in (a) case (ii). It is obvious that the condition for the case $2 \leq c < 3$ will be the same also.

(d)

It matters. If government is allowed to make a loss, we can make such mechanism. Let $a(r, n)$ be

$$a(\cdot, \cdot) = 0$$

This means that all of costs is covered by the government. In this case, it is one of Nash equilibria that every firm reports truely.

However, if government is allowed to make a profit only, not a loss, the range of c in which such mechanism exists does not get larger. It could be the case that the range of c is smaller than under the assumption of balancedness.