1.

Define as

$G(y) = \Pr(v \leq y)$: cdf of v

$f(v)$: bidding function

v_i : true value of the bidder i

w_i : the value the bidder i will pretend to have

Let b_i be i's bid, then

$b_i = f(w_i)$, or $w_i = f^{-1}(b_i)$

Thus Expected utility for the bidder is

$$
\begin{aligned}
EU_i(b|v_i, v_{-i}) &= E(v_i - b|b \text{ wins}) + E(-b|b \text{ loses}) \\
&= E(v_i|b \text{ wins}) - b \\
&= E[v_i|b > f(v_{-i})] - f(w_i) \\
&= E[v_i|f(w_i) > f(v_{-i})] - f(w_i) \qquad (\because b_i = f(w_i)) \\
&= E(v_i|w_i > v_{-i}) - f(w_i) \qquad (\because f(\cdot) \text{ is increasing function}) \\
&= v_i \cdot \Pr(w_i > v_{-i}) - f(w_i) \\
&= v_i \cdot G_{(w_i)}^{I-1} - f(w_i)
\end{aligned}
$$

Solve the maximization problem:

$$\max_w EU_i[f(w)|v, f_{-i}] = v \cdot G_{(w)}^{I-1} - f(w)$$

FOC is

$$\left.\frac{\partial}{\partial w} EU_i[f(w)|v, f_{-i}]\right|_{w=v} = 0$$

since $f(w) = f(v) \Leftrightarrow w = v$ by equilibrium condition.

$$
\begin{aligned}
\left.(I-1)vG_{(w)}^{I-2} \cdot G'_{(w)} - f(w)\right|_{w=v} &= 0 \\
(I-1)vG_{(v)}^{I-2} \cdot G'_{(v)} - f(v) &= 0 \\
\int_0^{\bar{v}} (I-1)vG_{(v)}^{I-2} \cdot G'_{(v)} dv &= \int_0^{\bar{v}} f(v) dv
\end{aligned}
$$

Since $f(0) = 0$, we have

$$f(\bar{v}) = (I-1) \int_0^{\bar{v}} vG_{(v)}^{I-2} \cdot G'_{(v)} dv$$

Let $G(y) = y$, then

$$
\begin{aligned}
f(\bar{v}) &= (I-1) \int_0^{\bar{v}} v \cdot v^{I-2} dv \\
&= (I-1) \int_0^{\bar{v}} v^{I-1} dv \\
&= \frac{I-1}{I} \bar{v}^I
\end{aligned}
$$

Therefore, the symmetric eqilibruim bid for All-Pay auction is

$$f(\bar{v}) = \frac{I-1}{I} \bar{v}^I$$

2.

Define as

$G(y) = \Pr(v \leq y)$: cdf of v

$f(v)$: bidding function

v_i : true value of the bidder i

w_i : the value the bidder i will pretend to have

Let b_i be i's bid, then

$b_i = f(w_i)$, or $w_i = f^{-1}(b_i)$

(a) First Price Auction

(i) Expected utility for the bidder

$$\begin{aligned}
EU_i(b|v_i, v_{-i}) &= E(v_i - b|b \text{ wins}) \\
&= E[v_i - b|b > f(v_{-i})] \\
&= E[v_i - f_{(w_i)}|f(w_i) > f(v_{-i})] \qquad (\because b_i = f(w_i)) \\
&= E[v_i - f_{(w_i)}|w_i > v_{-i}] \qquad (\because f(\cdot) \text{ is increasing function}) \\
&= (v_i - f_{(w_i)}) \cdot \Pr[w_i > v_{-i}] \\
&= (v_i - f_{(w_i)}) \cdot G_{(w_i)}^{I-1}
\end{aligned}$$

Solve the maximization problem:

$$\max_w EU_i[f(w)|v, f_{-i}] = (v - f_{(w)}) \cdot G_{(w)}^{I-1}$$

FOC is

$$\begin{aligned}
\frac{\partial}{\partial w} EU_i[f(w)|v, f_{-i}]\bigg|_{w=v} &= 0 \\
-f'_{(w)} \cdot G_{(w)}^{I-1} + (I-1)(v - f_{(w)}) \cdot G_{(w)}^{I-2} G'_{(w)}\bigg|_{w=v} &= 0 \\
-f'_{(v)} \cdot G_{(v)}^{I-1} + (I-1)(v - f_{(v)}) \cdot G_{(v)}^{I-2} G'_{(v)} &= 0 \\
f'_{(v)} G_{(v)}^{I-1} + (I-1) f_{(v)} G_{(v)}^{I-2} \cdot G'_{(v)} &= (I-1) v G_{(v)}^{I-2} \cdot G'_{(v)}
\end{aligned}$$

Note that $f(0) = 0$. Integrating both sides between 0 and $\bar{v}$ (the value for the bidder),

$$\begin{aligned}
f_{(\bar{v})} G_{(\bar{v})}^{I-1} &= \int_0^{\bar{v}} (I-1) v G_{(v)}^{I-2} \cdot G'_{(v)} dv \\
f(\bar{v}) &= \frac{I-1}{G_{(\bar{v})}^{I-1}} \int_0^{\bar{v}} v G_{(v)}^{I-2} \cdot G'_{(v)} dv
\end{aligned}$$

Let $G(y) = y$, then we obtain the symmetric eqilibruim bid for First Price Auction

$$\begin{aligned}
f(\bar{v}) &= \frac{I-1}{\bar{v}^{I-1}} \int_0^{\bar{v}} v^{I-1} dv \\
&= \frac{I-1}{\bar{v}^{I-1}} \cdot \frac{\bar{v}^I}{I} \\
&= \frac{I-1}{I} \bar{v}
\end{aligned}$$

(ii) Expected Revenue for the seller

Without loss of generality consider bidder 1 is the winner. Expected revenue for the seller is

$$\begin{aligned}
\text{ER} &= I \underset{v_1 > v_{-1}}{\int \cdots \int} \left[\left(\frac{I-1}{I} \right) v_1 \right] dv_I \cdots dv_1 \\
&= I \int_0^1 \int_0^{v_1} \cdots \int_0^{v_1} \left[\left(\frac{I-1}{I} \right) v_1 v_I \Big|_0^{v_1} \right] dv_{I-1} \cdots dv_1 \\
&= I \int_0^1 \left(\frac{I-1}{I} \right) v_1^I dv_1 \\
&= I \left(\frac{I-1}{I} \right) \frac{v^{I+1}}{I+1} \Big|_0^1 \\
&= \frac{I-1}{I+1}
\end{aligned}$$

(b) Second Price Auction

(i) Expected utility for the bidder

Define as

b_i : own bids

v_i : own true values

b^* : the highest bid by others

		i Bids v_i		i Bids b_i	
		Result	Utility	Result	Utility
1	$b_i < v_i < b^*$	Lose	0	Lose	0
2	$v_i < b_i < b^*$	Lose	0	Lose	0
3	$b_i < b^* < v_i$	Win	$v_i - b^* > 0$	Lose	0
4	$v_i < b^* < b_i$	Lose	0	Win	$v_i - b^* < 0$
5	$b^* < b_i < v_i$	Win	$v_i - b^* > 0$	Win	$v_i - b^* > 0$
6	$b^* < v_i < b_i$	Win	$v_i - b^* > 0$	Win	$v_i - b^* > 0$

Bidding the true value v_i weakly dominates bidding any other value b_i. Therefore,

$$f_{(\bar{v})} = \bar{v}$$

(ii) Expected Revenue for the seller

Consider bidder 1 is the winner and bidder 2 is the second highest bidder. In this case, the revenue for the seller will be v_2, and the expected revenue is the multiple integral over distribution of $v_1, \cdots, v_I$. This multiple integral should be multiplied by the number of all possible combinations of winners and runner ups, that is, $I(I-1)$. Expected revenue for the seller is

$$
\begin{aligned}
ER &= I(I-1) \int\limits_{v_1>v_2} \int\limits_{v_2>v_i} \cdots \int v_2 \; dv_I \cdots dv_1 \\
&= I(I-1) \int_0^1 \int_0^{v_1} \int_0^{v_2} \cdots \int_0^{v_2} v_2 \; dv_I \cdots dv_1 \\
&= I(I-1) \int_0^1 \int_0^{v_1} v_2^{I-1} \; dv_2 dv_1 \\
&= I(I-1) \int_0^1 \frac{v_1^I}{I} dv_1 \\
&= I(I-1) \cdot \frac{1}{I(I+1)} \\
&= \frac{I-1}{I+1}
\end{aligned}
$$

(c) All Pay Auction

(i) Expected utility for the bidder
As shown in question 1, the symmetric euqilibruim bid is

$$f(\bar{v}) = \frac{I-1}{I} \bar{v}^I$$

(ii) Expected Revenue for the seller

$$
\begin{aligned}
ER &= I \cdot E[f(v)] \\
&= I \cdot E\left[\frac{I-1}{I} v^I\right] \\
&= I \int_0^1 \frac{I-1}{I} v^I dv \\
&= \frac{I-1}{I+1}
\end{aligned}
$$

In all the cases of (a), (b) and (c), the expected revenues to the seller in the symmetric equilibrium are identical, as long as the item is always assigned to the person who values it the most. The expected revenue is always

$$ER = \frac{I-1}{I+1}$$

and thus it depends only on the number of bidders. Therefore this result agrees with the Revenue Equivalence Principle.

3. (a)

Denote t_i as buyer i's true value. Define s as buyer 1's pretending value. If $B_1(s) > B_2(t_2)$, buyer 1 wins. Thus,

$$Eu_1(s|t_1) = \int_{B_1(s)>B_2(t_2)} [t_1 - B_1(s)]\, dF_2(t_2)$$

Suppose that $B_1 = B_2 = B$. We are required to show that it should be that $F_1 = F_2$. Since $B_1 = B_2 = B$, $B_1(s) > B_2(t_2)$ is equivalent to $s > t_2$.

$$Eu_1(s|t_1) = \int_{s>t_2} [t_1 - B(s)]\, dF_2(t_2) = [t_1 - B(s)]\, F_2(s)$$

$$\frac{\partial Eu_1(s|t_1)}{\partial s} = [t_1 - B(s)]\, F_2'(s) - B'(s)F_2(s)$$

Since this equation should be 0 when $s = t_1$,

$$[t_1 - B(t_1)]\, F_2'(t_1) - B'(t_1)F_2(t_1) = 0$$

$$t_1 F_2'(t_1) = B(t_1)F_2'(t_1) + B'(t_1)F_2(t_1)$$

This holds for every $t_1 \in [0, 1]$. Taking integral between 0 and t,

$$tF_2(t) - \int_0^t F_2(u)du = B(t)F_2(t)$$

$$[t - B(t)]F_2(t) = \int_0^t F_2(u)du$$

Similarly

$$[t - B(t)]F_1(t) = \int_0^t F_1(u)du$$

Thus

$$[t - B(t)][F_1(t) - F_2(t)] = \int_0^t [F_1(u) - F_2(u)]du \quad \forall t \in [0, 1] \tag{1}$$

We claim that $F_1 = F_2$. To see this, define $G(t) = \int_0^t [F_1(u) - F_2(u)]du$. Note that G is smooth as well since F_1 and F_2 are smooth. Note also that $G'(t) = 0$ implies $F_1(t) = F_2(t)$. Clearly,

$$G(0) = \int_0^0 [F_1(u) - F_2(u)]du = 0$$
$$G(1) = [1 - B(1)][F_1(1) - F_2(1)] = 0$$

by (1). By the mean value theorem, there exists some $y \in (0, 1)$ such that $G'(y) = 0$. There are two cases we have to consider.

(i) $G'(y) = 0$ for all $y \in (0, 1)$

We have $F_1(y) = F_2(y)$ for all $y \in (0, 1)$, so we are done.

(ii) $G'(y) \neq 0$ for some y
Smoothness of G implies that there exist $y_1 < y_2$ such that $G'(y_1) = G'(y_2) = 0$ and $G'(y) > (<)\ 0$ for all $y \in (y_1, y_2)$.

$$\begin{aligned} G(y_2) &= \int_0^{y_2} G'(y)dy \\ &= \int_0^{y_1} G'(y)dy + \int_{y_1}^{y_2} G'(y)dy \\ &\underset{(<)}{>} \int_0^{y_1} G'(y)dy \\ &= G(y_1) \end{aligned}$$

However, from $G'(y_1) = G'(y_2) = 0$, we have $F_1(y_1) = F_2(y_1)$ and $F_1(y_2) = F_2(y_2)$, which implies $G(y_1) = G(y_2) = 0$ by (1). There is a contradiction.

Therefore, $B_1 = B_2$ implies $F_1 = F_2$. ∎

(b)
Since $B_1 \neq B_2$, there exists some interval T_1 such that for all $t_1 \in T_1$, $B_1(t_1) \neq B_2(t_1)$. Without loss of generality $B_1(t_1) > B_2(t_1)$. For each $t_1 \in T_1$, there exists $z(t_1)$ such that

$$B_1(t_1) = B_2(z)$$

Thus for all t_2 such that $t_1 < t_2 < z(t_1)$,

$$B_1(t_1) > B_2(t_2)$$

since B_2 is strictly increasing. So for such (t_1, t_2), buyer 1 wins even though $t_2 > t_1$.

$$\begin{aligned} \Pr(\text{bidder with the lower value wins}) &\geq \int_{t_1 \in T_1} \int_{t_1}^{z(t_1)} 1\ dF_2(t_2) dF_1(t_1) \\ &> 0 \end{aligned}$$

4. (a)
The strategies at the Nash equilibrium in which 1 proposes $(1, 49)$ on the first day and 2 accepts are

(I) the strategy of player 1:
On odd days, 1 proposes "1 for himself, and everything else for 2."
On even days, 1 rejects all offers.

(II) the strategy of player 2 :
On odd days, 2 accepts offers which give 1 to player 1 and everything else to 2, and rejects others.
On even days, 2 offers "everything for himself, and 0 for 1."

In (I,II), 1 proposes $(1, 49)$ and 2 accepts and the game ends. To see that (I,II) is a Nash equlibrium, if 1 changes his strategy, he can get at most 1, since 2 rejects all offers which give player 1 more than 1 and proposes only 0 to player 1. If 2 changes his strategy, he cannot get more than 49.

(b)

The strategies at the Nash equilibrium in which 1 proposes $(49, 1)$ on the first day and 2 accepts are

(I) the strategy of player 1:
On odd days, 1 proposes "1 for player 2, and everything else for himself."
On even days, 1 rejects all offers.

(II) the strategy of player 2 :
On odd days, 2 accepts all offers.
On even days, 2 offers "everything for himself, and 0 for 1."

In (I,II), 1 proposes $(49, 1)$ and 2 accepts and the game ends. To see that (I,II) is a Nash equlibrium, if 2 changes his strategy, he can get at most 1, since 1 rejects all offers and proposes only 1 to player 2 every odd day. If 1 changes his strategy, he cannot get more than 49. There are many Nash equilibrium in which 1 proposes $(49, 1)$ on the first day and 2 accepts. The above is one of them.

(c)

Yes, there exist many such Nash equilibria in which the players first come to agreement on day 5. One trivial example is that they play the following strategies.

(I) the strategy of player 1:
On odd days, 1 proposes "1 for player 2, and everything else for himself."
On even days, 1 rejects all offers.

(II) the strategy of player 2 :
On odd days but the 5th day, 2 rejects all offers.
On the 5th day, 2 accepts all offers.
On even days, 2 offers "everything for himself, and 0 for 1."

In (I,II), 1 proposes (49,1) on the first day and 2 rejects. 2 proposes (0,48) on the second day and 1 rejects. 1 proposes (45,1) on the third day and 2 rejects. 2 proposes (0,44) on the 4th day and 1 rejects. 1 proposes (41,1) on the 5th day and 2 accepts, and the game ends. To see that (I,II) is a Nash equilibrium, if 2 changes his strategy, he can get at most 1, since 1 rejects all offers and proposes only 1 to player 2 every odd day. If 1 changes his strategy, he cannot get more than 41, since 2 rejects all offers made on any day other than the 5th day and proposes only 0 to player 1 every even day.

5. (a)

The answer is NO.
By backward induction, consider the last day first.
2 will accept any offer since he gets more than or equal to 1 if he accepts, but 0 if he rejects.
Given this, 1 will offer $(49, 1)$.

Consider any even day.
1 will accept $(50, 0)$ but never accept $(48, 2)$ or less favorable offers.
Given this, 2 will never offer $(50, 0)$, since he gets 0 if he offers $(50, 0)$,
but he can get at least 1 on the last day.
Therefore, 2 will offer any other than $(50, 0)$ and 1 will reject.

Consider any odd day.

2 will accept any offer favorable than $(49, 1)$.

Given this, 1 will never offer $(47, 3)$ or less favorable offers,

since he can get at least 49 on the last day.

So 1 will offer $(49, 1)$.

1 will offer $(49, 1)$ on the first day also. Therefore, it cannot be a Subgame Perfect Equilibrium that 1 proposes $(1, 49)$ on the first day.

(b)

The answer is YES.

With the same reason as in (a), 1 will offer $(49, 1)$ on the first day. Think about 2's reasoning.

2 knows that 1 will offer $(49, 1)$ every odd day, and the last day is odd day.

2 knows that there is no subgame perfect equilibrium in which the players agree on even days.

2 concludes that he can get at most 1 on any day.

Therefore, accepting 1's offer on the first day is one of the best strategies for 2. One of Subgame Perfect Equilibria supporting this outcome is a profile of two players' strategies :

1 always proposes $(49, 1)$, accepts $(50, 0)$ and rejects all other offers.

2 always accepts all offers and proposes $(48, 2)$ on any odd days.

(c)

The answer is YES.

As described in (b), 2's best strategy is accepting any of $(49, 1)$ offers and getting 1. 1 will offer $(49, 1)$ on the 5th day also, so 2 can reject first two offers and accept day 5 offer $(49, 1)$. One of Subgame Perfect Equilibria supporting this outcome is a profile of two players' strategies :

1 always proposes $(49, 1)$, accepts $(50, 0)$ and rejects all other offers.

2 rejects $(49, 1)$ and accepts all others on the 1st and the 3rd day.

2 accepts all offers on the other days and proposes $(48, 2)$ on any odd days.

Note first that there are many other Subgame Perfect Equilibria supporting this outcome. For example, 2 can propose $(46, 4)$ in another equilibrium. Note also that there are many Subgame Perfect Equilibria in which 2 rejects some of offers $(49, 1)$ and accepts on any odd day for the first time. The reason why there are too many Subgame Perfect Equilibria is that there are ties among payoffs resulting from different profiles of strategies. This is the difference from the game we analyzed in the class. If there is no tie, then only one Subgame Perfect Equilibrium exists. Note that in any Subgame Perfect Equilibria of this game, the following should be ture.

1 always proposes $(49, 1)$, accepts $(50, 0)$ and rejects all other offers.

2 never offers $(50, 0)$.

2 never rejects $(49, 1)$ on the last day.

The resulting payoff is $(49, 1)$.