5. (a)

Payoffs are as follows.

		COL	
		C	D
ROW	C	3 3	0 5
	D	5 0	1 1

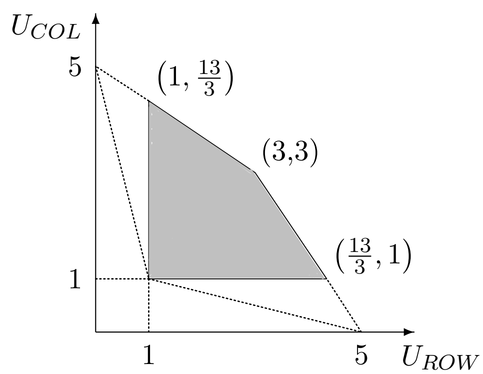
In infinitely repeated game, there are infinitely many Subgame Perfect Equilibria, as Folk Theorem says. For any such equilibrium, the minimum payoff that one can obtain by deviating from the equilibrium is

$$1 + \delta + \delta^2 + \cdots = \frac{1}{1-\delta}$$

Suppose that E is an equilibrium. Then COL should be able to obtain at least $\frac{1}{1-\delta}$ in E. If not, COL can obtain more by deviating, which means that E is not an equilibrium. In fact, there exists an equlibrium in which COL obtain $\frac{1}{1-\delta}$. It is that both players always D. So the lowest payoff that COL can obtain is

$$\frac{1}{1-\delta}$$

(b)



Any point in the shaded area can be achievable as discounted average payoffs of both players with some $\delta \in (0, 1]$ and a set of strategy profiles that is a Subgame Perfect Equilibrium. Therefore, if we can pick δ, the highest discounted average payoff that ROW can obtain is $\frac{13}{3}$. But when δ is fixed, the answer will be complicated. Consider some Grim-Trigger strategy profile in which players follow suggested actions, but when any of them deviates, both play D thereafter.

(i) GT : both play C

This gives each player $\frac{3}{1-\delta}$. If one deviates at first period, he gets $5 + \frac{\delta}{1-\delta}$. So this is an equilibrium if $\frac{3}{1-\delta} \geq 5 + \frac{\delta}{1-\delta}$, or $\delta \geq 0.5$. Since $\delta > 0.5$, this is an equilibrium.

(ii) GT : ROW plays D, C and COL plays C, D and repeat this.

If both follow suggestion,

$$U_{ROW} = \frac{5}{1-\delta^2}$$

$$U_{COL} = \frac{5\delta}{1-\delta^2}$$

It is obvious that ROW doesn't want to deviate unless COL wants to deviate. COL doensn't want to deviate, if $\frac{5\delta}{1-\delta^2}$ is greater than $\frac{1}{1-\delta}$, which is the payoff COL can get when he deviates at first period. This condition is $\delta \geq 0.25$.

(iii) GT : ROW plays repitition of D, D, C and COL plays repetition of C, C, D.

$U_{ROW} = \dfrac{5+5\delta}{1-\delta^3}$

$U_{COL} = \dfrac{5\delta^2}{1-\delta^3}$

The condition that this is an equilibrium is $\delta \geq \frac{1+\sqrt{17}}{4} \approx 0.6404$.

(iv) GT : ROW plays repitition of D, D, D, C and COL plays repetition of C, C, C, D.

$U_{ROW} = \dfrac{5+5\delta+5\delta^2}{1-\delta^4}$

$U_{COL} = \dfrac{5\delta^3}{1-\delta^4}$

The condition that this is an equilibrium is $\delta \geq 0.8689$, approximately.

(v) GT : ROW plays repitition of D, D, D, D, C and COL plays repetition of C, C, C, C, D.

$U_{ROW} = \dfrac{5+5\delta+5\delta^2+5\delta^3}{1-\delta^5}$

$U_{COL} = \dfrac{5\delta^4}{1-\delta^5}$

The condition that this is an equilibrium is $\delta \geq 1$.

(vi) GT : ROW plays repitition of D, C and COL plays C forever.

$U_{ROW} = \dfrac{5+3\delta}{1-\delta^2}$

$U_{COL} = 0 + 3\delta + 0 \cdot \delta^2 + 3\delta^3 + \cdots = \dfrac{3\delta}{1-\delta^2}$

The most profitable deviation of COL is to play D at second period.

The condition that this deviation is worse than GT is $\delta \geq \frac{1+\sqrt{33}}{8} \approx 0.8431$.

(vii) GT : ROW repeats D, D, C and COL plays C forever.

$U_{ROW} = \dfrac{5+5\delta+3\delta^2}{1-\delta^3}$

$U_{COL} = \dfrac{3\delta^2}{1-\delta^3}$

The condition that this is an equilibirum is $\delta \geq 1$.

There are too many equilibria to consider. If we consider payoffs in above seven equilibria, we have to compare U_{ROW} of each strategy profile. We can easily verify that $U_{ROW(\text{v})} > U_{ROW(\text{iv})} > U_{ROW(\text{iii})} > U_{ROW(\text{ii})}$ and that $U_{ROW(\text{vii})} > U_{ROW(\text{vi})}$. So we only need to compare some pairs of them.

(1) $U_{ROW(\text{i})} = \dfrac{3}{1-\delta} \leq U_{ROW(\text{ii})} = \dfrac{5+5\delta}{1-\delta^3}$ when $\delta \leq \dfrac{2}{3}$

(2) $U_{ROW(\text{i})} \leq U_{ROW(\text{iii})}$ for any $\delta \leq 1.2153$

So $U_{ROW(\text{i})}$ cannot be maximum if δ is such that (iii) can be an equilibrium.

(3) $U_{ROW(\text{i})} \leq U_{ROW(\text{vi})}$ for any δ

(4) $U_{ROW}(\text{ii}) \le U_{ROW}(\text{vi})$ if $\delta \le \dfrac{-3+\sqrt{33}}{6} \approx 0.4574$

(5) $U_{ROW}(\text{iii}) \le U_{ROW}(\text{vi})$ if $\delta \le \sqrt{\frac{2}{3}} \approx 0.8165$

So we conclude that among above seven strategies,

range of δ	equilbrium that maximizes U_{ROW}	highest U_{ROW}
$0.5 \le \delta \le 0.6404$	(ii)	$\dfrac{5}{1-\delta^2}$
$0.6404 \le \delta \le 0.8431$	(iii)	$\dfrac{5+5\delta}{1-\delta^3}$
$\delta \ge 0.8431$	(vi)	$\dfrac{5+3\delta}{1-\delta^2}$

But note that there might be some equilibria in which U_{ROW} can be larger. In fact, the following strategy profile is an equilibrum if $\delta \ge 0.9405$.

(viii) GT : ROW repeats D, D, C, D, C and COL plays C forever.

$$U_{ROW} = \frac{5+5\delta+3\delta^2+5\delta^3+3\delta^4}{1-\delta^5}$$

This U_{ROW} is greater than any U_{ROW} of (i)-(vi) when $\delta \ge 0.9405$. (This is less than $U_{ROW}(\text{vii})$ however.) We can find more. Therefore, if δ is given, we can find some strategy profile that is a Subgame Perfect Equliibrium and that maximizes ROW's expected utility. Note that ROW's highest utility for given δ is less than $\frac{13}{3(1-\delta)}$ unless $\delta = 1$.

The highest payoff ROW can obtain changes as δ changes. It is a non-decreasing function of δ because as δ increases, the present value of income stream rises and also the set of strategy profiles that can be sustained as a Subgame Perfect Equlibrium gets larger.

6. (a)

By one-shot deviation principle, we only need to check one shot deviation . Let both players use Grim Trigger strategy with $Q_1 = Q_2 = 0.25$. If nobody deviates, both get a stream of profit $\frac{1}{8}$ each period. Then if 2 deviates, the most profitable Q_2 is $\frac{3}{8}$, and the profit at that time is $\frac{9}{64}$. Once he deviates, 2 will get Cournot profit $\frac{1}{9}$ thereafter. So 2 will not deviate if

$$\frac{1}{8} \cdot \frac{1}{1-\delta} \ge \frac{9}{64} + \frac{1}{9} \cdot \frac{\delta}{1-\delta}$$

or

$$\delta \ge \frac{9}{17}$$

(b)

Only if $\delta = 0$, there is a unique Subgame Pefect Equilibrium in which both produces $Q_1 = Q_2 = \frac{1}{3}$ in every period.

To see this, for given $\delta > 0$, find a Subgame Perfect Equilibrium. Suppose that GT strategy with $Q_1 = Q_2 = q$ is a Subgame Perfect Equilibrium. If nobody deviates, both get a stream of profit

$$\text{profit} = q(1-2q)$$

each period. If 2 deviates, the most profitable Q_2 is

$$Q_2 = \arg\max Q_2(1 - Q_2 - q)$$

or

$$Q_2 = \frac{1-q}{2}$$

So the profit at that time is

$$\text{profit}_0 = \left(\frac{1-q}{2}\right)^2$$

Once he deviates, he will get $\frac{1}{9}$ thereafter. So, by one-shot deviation principle, 2 will not deviate if

$$\frac{q(1-2q)}{1-\delta} \geq \left(\frac{1-q}{2}\right)^2 + \frac{\delta}{9(1-\delta)}$$

Define the difference as f

$$\begin{aligned}
f(\delta, q) &= \frac{q(1-2q)}{1-\delta} - \left(\frac{1-q}{2}\right)^2 - \frac{\delta}{9(1-\delta)} \\
&= \frac{1}{36(1-\delta)}\left[36q(1-2q) - 9(1-\delta)(1-q)^2 - 4\delta\right] \\
&= \frac{1}{36(1-\delta)}\left[-(81-9\delta)q^2 + (54-18\delta)q - 9 + 5\delta\right] \\
&= \frac{1}{36(1-\delta)}\left[-(81-9\delta)\left(q - \frac{27-9\delta}{81-9\delta}\right)^2 + \frac{(27-9\delta)^2}{81-9\delta} - 9 + 5\delta\right] \\
&= -\frac{9-\delta}{4(1-\delta)}\left(q - \frac{3-\delta}{9-\delta}\right)^2 + \frac{1}{36(1-\delta)}\left[\frac{(9-3\delta)^2}{9-\delta} - 9 + 5\delta\right] \\
&= -\frac{9-\delta}{4(1-\delta)}\left(q - \frac{3-\delta}{9-\delta}\right)^2 + \frac{1}{36(1-\delta)(9-\delta)}\left[(9-3\delta)^2 + (9-\delta)(-9+5\delta)\right] \\
&= -\frac{9-\delta}{4(1-\delta)}\left(q - \frac{3-\delta}{9-\delta}\right)^2 + \frac{\delta^2}{9(1-\delta)(9-\delta)}
\end{aligned}$$

Take $q = \frac{3-\delta}{9-\delta}$, then

$$f(\delta, q) = \frac{\delta^2}{9(1-\delta)(9-\delta)} > 0$$

as long as $\delta > 0$. So, 2 will not deviate from such q, and the same holds for 1, too. Thus, GT with $Q_1 = Q_2 = q$ is a Subgame Perfect Equilibrium for given δ. Therefore, as long as $\delta > 0$, Subgame Perfect Equilibrium is not unique.

When $\delta = 0$, both players put whole weight on today's payoff only, so they play one-shot game in every period. So the only Subgame Perfect Equilibrium is $Q_1 = Q_2 = \frac{1}{3}$ in every period.