ECON 271B assignment #2

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Yang, Yong Hyeon

Dynamic Programming/Estimation

(1) Setting up Dynamic Programming Problem

Since ε_t's are iid, the present discounted value of firm's future profits is

$$V(a_t, \varepsilon_{0t}, \varepsilon_{1t}) = \max_{i_t, i_{t+1}, i_{t+2}, \cdots} \left\{ \Pi(a_t, i_t, \varepsilon_{0t}, \varepsilon_{1t}) + \sum_{s=t+1}^{\infty} \beta^{s-t} E[\Pi(a_s, i_s, \varepsilon_{0s}, \varepsilon_{1s}) | a_t, i_t] \right\}$$

Writing this equation recursively, we have the following dynamic programming problem.

$$V(a_t, \varepsilon_{0t}, \varepsilon_{1t}) = \max_{i_t} \left\{ \Pi(a_t, i_t, \varepsilon_{0t}, \varepsilon_{1t}) + \beta E[V(a_{t+1}, \varepsilon_{0t+1}, \varepsilon_{1t+1}) | a_t, i_t] \right\}$$

(2) Rust's Alternative Specification

Define $\overline{V}_0$ and $\overline{V}_1$ as follows.

$$\begin{aligned} \overline{V}_0(a_t) &= \theta_1 a_t + \beta E[V(a_{t+1}, \varepsilon_{0t+1}, \varepsilon_{1t+1}) | a_t, i_t = 0] \\ \overline{V}_1(a_t) &= R + \beta E[V(a_{t+1}, \varepsilon_{0t+1}, \varepsilon_{1t+1}) | a_t, i_t = 1] \end{aligned}$$

Using $\overline{V}_0$ and $\overline{V}_1$, we can rewrite V as

$$V(a_t, \varepsilon_{0t}, \varepsilon_{1t}) = \max \left\{ \overline{V}_0(a_t) + \varepsilon_{0t}, \overline{V}_1(a_t) + \varepsilon_{1t} \right\}$$

So $\overline{V}_0$ and $\overline{V}_1$ would be

$$\begin{aligned} \overline{V}_0(a_t) &= \theta_1 a_t + \beta E \left[\max \left\{ \overline{V}_0(a_{t+1}) + \varepsilon_{0t+1}, \overline{V}_1(a_{t+1}) + \varepsilon_{1t+1} \right\} \middle| a_t, i_t = 0 \right] \\ \overline{V}_1(a_t) &= R + \beta E \left[\max \left\{ \overline{V}_0(a_{t+1}) + \varepsilon_{0t+1}, \overline{V}_1(a_{t+1}) + \varepsilon_{1t+1} \right\} \middle| a_t, i_t = 1 \right] \end{aligned}$$

Since ε_t's are Logit errors,

$$\begin{aligned} \overline{V}_0(a_t) &= \theta_1 a_t + \beta E \left[0.5772 + \log \left(e^{\overline{V}_0(a_{t+1})} + e^{\overline{V}_1(a_{t+1})} \right) \middle| a_t, i_t = 0 \right] \\ \overline{V}_1(a_t) &= R + \beta E \left[0.5772 + \log \left(e^{\overline{V}_0(a_{t+1})} + e^{\overline{V}_1(a_{t+1})} \right) \middle| a_t, i_t = 1 \right] \end{aligned}$$

This expectation is easy to calculate given that a_t has the following first order Markov process.

$$a_{t+1} = \begin{cases} \min\{5, a_t + 1\} & \text{if } i_t = 0 \\ 1 & \text{if } i_t = 1 \end{cases}$$

For example,

$$\begin{aligned} \overline{V}_0(3) &= 3\theta_1 + \beta \left[0.5772 + \log \left(e^{\overline{V}_0(4)} + e^{\overline{V}_1(4)} \right) \right] \\ \overline{V}_1(3) &= R + \beta \left[0.5772 + \log \left(e^{\overline{V}_0(1)} + e^{\overline{V}_1(1)} \right) \right] \end{aligned}$$

(3) Preliminaries

Using value function iteration method, we can get $\overline{V}_0(a_t)$ and $\overline{V}_1(a_t)$ at the five values of a_t as follows, when the parameter values are such that $\theta_1 = -1$ and $R = -3$.

a_t	$\overline{V}_0(a_t)$	$\overline{V}_1(a_t)$
1	−10.1692	−11.4027
2	−11.5172	−11.4027
3	−12.6576	−11.4027
4	−13.7106	−11.4027
5	−14.7106	−11.4027

When a_t=2, $\overline{V}_0(a_t) = -11.5172$ and $\overline{V}_1(a_t) = -11.4027$, so the firm is indifferent between replacing its machine or not if

$$\overline{V}_0(a_t) + \varepsilon_{0t} = \overline{V}_1(a_t) + \varepsilon_{1t} \iff \varepsilon_{0t} - \varepsilon_{1t} = \overline{V}_1(a_t) - \overline{V}_0(a_t) = 0.1145$$

The probability that this firm replaces its machine is

$$\Pr\left(\overline{V}_1(a_t) + \varepsilon_{1t} > \overline{V}_0(a_t) + \varepsilon_{0t}\right) = \frac{e^{\overline{V}_1(a_t)}}{e^{\overline{V}_0(a_t)} + e^{\overline{V}_1(a_t)}} \approx 0.5286$$

If a firm has $a_t = 4$, $\varepsilon_{0t} = 1$ and $\varepsilon_{1t} = -1.5$, then

$$\begin{aligned} \overline{V}_0(a_t) + \varepsilon_{0t} &= -12.7106 \\ \overline{V}_1(a_t) + \varepsilon_{1t} &= -12.9027 \end{aligned}$$

So the PDV of future profits for this firm is

$$V(a_t, \varepsilon_{0t}, \varepsilon_{1t}) = \max\{\overline{V}_0(a_t) + \varepsilon_{0t}, \overline{V}_1(a_t) + \varepsilon_{1t}\} = -12.7106$$

(4) MLE

For any $\theta = (\theta_1, R)$, we can get value functions $\overline{V}_0$ and $\overline{V}_1$. Then the probability that the firm with a_j replaces the machine is given by

$$p(a_j) \equiv \Pr\left(\overline{V}_1(a_j) + \varepsilon_{1j} > \overline{V}_0(a_j) + \varepsilon_{0j}\right) = \frac{e^{\overline{V}_1(a_j)}}{e^{\overline{V}_0(a_j)} + e^{\overline{V}_1(a_j)}}$$

An individual likelihood function is then

$$f(a_j, i_j; \theta) = p(a_j)^{i_j}[1 - p(a_j)]^{1-i_j}$$

ML estimator $\widehat{\theta}$ maximizes a joint log-likelihood

$$\max_{\theta} \quad l(a, i; \theta) = \frac{1}{N}\sum_{j=1}^{N} \log f(a_j, i_j; \theta)$$

	θ_1	R
MLE	-1.1484	-4.4464
(s.e.)	(0.0749)	(0.3210)

a_t	$\overline{V}_0(a_t)$	$\overline{V}_1(a_t)$
1	-16.9954	-19.1213
2	-18.6826	-19.1213
3	-20.0337	-19.1213
4	-21.2499	-19.1213
5	-22.3983	-19.1213

(5) Standard Error

ML estimator $\widehat{\theta}$ has the following asymptotic distribution

$$\sqrt{N}\left(\widehat{\theta}-\theta_0\right) \xrightarrow{d} N\left(0,[i(\theta_0)]^{-1}\right)$$

where $i(\theta_0)$ denotes the information matrix of the likelihood function. So its large sample variance is given by

$$Var(\widehat{\theta}) = \frac{1}{N}\left(E\left[\frac{\partial \log f(a_j,i_j;\theta_0)}{\partial\theta}\frac{\partial \log f(a_j,i_j;\theta_0)}{\partial\theta'}\right]\right)^{-1}$$

We can approximate this by

$$\begin{aligned}\widehat{V} &= \frac{1}{N}\left(\frac{1}{N}\sum_{j=1}^{N}\frac{\partial \log f(a_j,i_j;\widehat{\theta})}{\partial\theta}\frac{\partial \log f(a_j,i_j;\widehat{\theta})}{\partial\theta'}\right)^{-1} \\ &= \left(\sum_{j=1}^{N}\frac{\partial \log f(a_j,i_j;\widehat{\theta})}{\partial\theta}\frac{\partial \log f(a_j,i_j;\widehat{\theta})}{\partial\theta'}\right)^{-1}\end{aligned}$$

The partial derivatives should be numerically computed. Perturb θ around $\widehat{\theta}$ by a little and get a log-likelihood, then we can approximate its derivative by its slope. Standard errors are reported in the top table.

(6) Modification of the Model

(a) There would be no change in any expression except for a transition matrix. A transition matrix of a_t in the case of $i_t = 0$ would change as

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} \lambda & 1-\lambda & 0 & 0 & 0 \\ 0 & \lambda & 1-\lambda & 0 & 0 \\ 0 & 0 & \lambda & 1-\lambda & 0 \\ 0 & 0 & 0 & \lambda & 1-\lambda \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

while that in the case of $i_t = 1$ does not change. So the value functions $\overline{V}_0$ and $\overline{V}_1$ would depend on λ as well as θ_1 and R. This ensures that λ is empirically identified. If different λ is used, we will have different value functions. As λ is greater, value functions would be greater too, since then machines are less likely to break down. As in (4), we can use ML estimation to find θ_1, R, and λ that maximize a joint log-likelihood. Since the data are a random sample of machines, ML estimator is consistent, and thus λ can be consistently estimated.

(b) We can find two different sets of value functions for θ_{1A} and θ_{1B}. The dynamic programming problems for both types of firms are the same as before, since there is a difference only in parameters between two types, not in structures such as profit functions or transition matrices. But the likelihood function would change. Assume first that we have data on types of firms. Let $s_j = 1$ if firm j has θ_{1A} and $s_j = 0$ otherwise. Once we get value functions corresponding to θ_{1A} and θ_{1B}, we can compute the probability that a firm with a_j and s_j replaces the machine as follows.

$$\begin{aligned} p_A(a_j) &\equiv \Pr\left(\overline{V}_{1A}(a_j) + \varepsilon_{1j} > \overline{V}_{0A}(a_j) + \varepsilon_{0j}\right) = \frac{e^{\overline{V}_{1A}(a_j)}}{e^{\overline{V}_{0A}(a_j)} + e^{\overline{V}_{1A}(a_j)}} \\ p_B(a_j) &\equiv \Pr\left(\overline{V}_{1B}(a_j) + \varepsilon_{1j} > \overline{V}_{0B}(a_j) + \varepsilon_{0j}\right) = \frac{e^{\overline{V}_{1B}(a_j)}}{e^{\overline{V}_{0B}(a_j)} + e^{\overline{V}_{1B}(a_j)}} \end{aligned}$$

An individual likelihood function is then

$$\begin{aligned} f(a_j, i_j, s_j; \theta_1, R) &= I(s_j = 1) \cdot p_A(a_j)^{i_j}[1 - p_A(a_j)]^{1-i_j} + I(s_j = 0) \cdot p_B(a_j)^{i_j}[1 - p_B(a_j)]^{1-i_j} \\ &= s_j p_A(a_j)^{i_j}[1 - p_A(a_j)]^{1-i_j} + (1 - s_j) \cdot p_B(a_j)^{i_j}[1 - p_B(a_j)]^{1-i_j} \end{aligned}$$

Again ML estimator maximizes a joint log-likelihood

$$\max_{\theta_1, R} \quad l(a, i, s; \theta_1, R) = \frac{1}{N} \sum_{j=1}^{N} \log f(a_j, i_j, s_j; \theta_1, R)$$

Now assume that firms' types are unobservable to econometricians. We have to form a likelihood function using α. α is a parameter to be estimated. An individual likelihood function is

$$\begin{aligned} f(a_j, i_j; \alpha, \theta_1, R) &= \Pr(s_j = 1) \cdot p_A(a_j)^{i_j}[1 - p_A(a_j)]^{1-i_j} + \Pr(s_j = 0) \cdot p_B(a_j)^{i_j}[1 - p_B(a_j)]^{1-i_j} \\ &= \alpha p_A(a_j)^{i_j}[1 - p_A(a_j)]^{1-i_j} + (1 - \alpha) \cdot p_B(a_j)^{i_j}[1 - p_B(a_j)]^{1-i_j} \end{aligned}$$

ML estimator maximizes a joint log-likelihood

$$\max_{\alpha, \theta_1, R} \quad l(a, i; \alpha, \theta_1, R) = \frac{1}{N} \sum_{j=1}^{N} \log f(a_j, i_j; \alpha, \theta_1, R)$$

(c) Everything is the same, except for a slight change in the joint log-likelihood function. This is because ε_t's are iid and the law of motion for a_t is deterministic. For example, in the case we do not observe firms' types, ML estimator maximizes the following joint log-likelihood.

$$\max_{\alpha, \theta_1, R} \quad l(a, i; \alpha, \theta_1, R) = \frac{1}{N} \frac{1}{T} \sum_{j=1}^{N} \sum_{t=1}^{T} \log f(a_{jt}, i_{jt}; \alpha, \theta_1, R)$$

(d) In this case, we have to set up a different dynamic programming problem. We can think of θ_1 as a state variable, not as a parameter. There is a law of motion for θ_1 given as follows.

$$\theta_{1t+1} = \begin{cases} \theta_{1t} & \text{if } i_t = 0 \\ \left.\begin{array}{ll} \theta_{1A} & \text{w.p. } \alpha \\ \theta_{1B} & \text{w.p. } 1-\alpha \end{array}\right\} & \text{if } i_t = 1 \end{cases}$$

Then the current period profit using a machine of age a_t and type θ_{1t} is given by

$$\Pi(a_t, i_t, \theta_{1t}, \varepsilon_{0t}, \varepsilon_{1t}) = \begin{cases} \theta_{1t} a_t + \varepsilon_{0t} & \text{if } i_t = 0 \\ R + \varepsilon_{1t} & \text{if } i_t = 1 \end{cases}$$

The initial Bellman equation is defined using the above state variables and functions.

$$V(a_t, \theta_{1t}, \varepsilon_{0t}, \varepsilon_{1t}) = \max_{i_t} \left\{ \Pi(a_t, i_t, \theta_{1t}, \varepsilon_{0t}, \varepsilon_{1t}) + \beta E[V(a_{t+1}, \theta_{1t+1}, \varepsilon_{0t+1}, \varepsilon_{1t+1}) | a_t, i_t, \theta_{1t}] \right\}$$

Here we can use Rust's alternative specification, too. Now there are three types of value functions.

$$\begin{aligned} \overline{V}_{0A}(a_t) &= \theta_{1A} a_t + \beta E[V(a_{t+1}, \theta_{1t+1}, \varepsilon_{0t+1}, \varepsilon_{1t+1}) | a_t, i_t = 0, \theta_{1t} = \theta_{1A}] \\ \overline{V}_{0B}(a_t) &= \theta_{1B} a_t + \beta E[V(a_{t+1}, \theta_{1t+1}, \varepsilon_{0t+1}, \varepsilon_{1t+1}) | a_t, i_t = 0, \theta_{1t} = \theta_{1B}] \\ \overline{V}_1(a_t) &= R + \beta E[V(a_{t+1}, \theta_{1t+1}, \varepsilon_{0t+1}, \varepsilon_{1t+1}) | a_t, \theta_{1t}, i_t = 1] \end{aligned}$$

V can be replaced by an expression using $\overline{V}_{0A}, \overline{V}_{0B}$ and $\overline{V}_1$.

$$V(a_{t+1}, \cdots) = \begin{cases} \max\left\{ \overline{V}_{0A}(a_{t+1}) + \varepsilon_{0t+1}, \overline{V}_1(a_{t+1}) + \varepsilon_{1t+1} \right\} & \text{if } i_t = 0, \theta_{1t} = \theta_{1A} \\ \max\left\{ \overline{V}_{0B}(a_{t+1}) + \varepsilon_{0t+1}, \overline{V}_1(a_{t+1}) + \varepsilon_{1t+1} \right\} & \text{if } i_t = 0, \theta_{1t} = \theta_{1B} \\ \max\left\{ \alpha \overline{V}_{0A}(a_{t+1}) + (1-\alpha)\overline{V}_{0B}(a_{t+1}) + \varepsilon_{0t+1}, \overline{V}_1(a_{t+1}) + \varepsilon_{1t+1} \right\} & \text{if } i_t = 1 \end{cases}$$

Then,

$$\begin{aligned} \overline{V}_{0A}(a_t) &= \theta_{1A} a_t + \beta E\left[\max\left\{ \overline{V}_{0A}(a_{t+1}) + \varepsilon_{0t+1}, \overline{V}_1(a_{t+1}) + \varepsilon_{1t+1} \right\} \Big| a_t, i_t = 0 \right] \\ \overline{V}_{0B}(a_t) &= \theta_{1B} a_t + \beta E\left[\max\left\{ \overline{V}_{0B}(a_{t+1}) + \varepsilon_{0t+1}, \overline{V}_1(a_{t+1}) + \varepsilon_{1t+1} \right\} \Big| a_t, i_t = 0 \right] \\ \overline{V}_1(a_t) &= R + \beta E\left[\max\left\{ \alpha \overline{V}_{0A}(a_{t+1}) + (1-\alpha)\overline{V}_{0B}(a_{t+1}) + \varepsilon_{0t+1}, \overline{V}_1(a_{t+1}) + \varepsilon_{1t+1} \right\} \Big| a_t, i_t = 1 \right] \end{aligned}$$

Use the Logit error assumption to rewrite the functions as

$$\begin{aligned} \overline{V}_{0A}(a_t) &= \theta_{1A} a_t + \beta E\left[0.5772 + \log\left(e^{\overline{V}_{0A}(a_{t+1})} + e^{\overline{V}_1(a_{t+1})} \right) \Big| a_t, i_t = 0 \right] \\ \overline{V}_{0B}(a_t) &= \theta_{1B} a_t + \beta E\left[0.5772 + \log\left(e^{\overline{V}_{0B}(a_{t+1})} + e^{\overline{V}_1(a_{t+1})} \right) \Big| a_t, i_t = 0 \right] \\ \overline{V}_1(a_t) &= R + \beta E\left[0.5772 + \log\left(e^{\alpha \overline{V}_{0A}(a_{t+1}) + (1-\alpha)\overline{V}_{0B}(a_{t+1})} + e^{\overline{V}_1(a_{t+1})} \right) \Big| a_t, i_t = 1 \right] \end{aligned}$$

The rest of the procedure would be the same, such as iterations on value function and ML estimation. One more complication I did not adopt here is that each type of machine might have different realizations of ε_t's. While it does not require much modification of the above procedure, it is not reasonable that a firm sees e_{0At+1} and e_{0Bt+1} before replacing the machine.

(7) Hotz and Miller Algorithm

We have the following problem.

$$\overline{P}(\cdot;\theta) = g\Big(h(\widehat{P}(\cdot);\theta);\theta\Big)$$

Use a consistent estimator $\widehat{P}(\cdot)$ for $P(\cdot;\theta_0)$, then $\overline{P}(\cdot;\theta)$ would also be consistent if θ is consistent for θ_0. We can construct a consistent estimator $\widehat{p}(a)$ for the probability that a firm with a replaces the machine as

$$\widehat{p}(a) = \frac{\#\text{replacements at } a}{\#\text{firms with } a} = \frac{\sum_{j=1}^{N} I(a_j = a, i_j = 1)}{\sum_{j=1}^{N} I(a_j = a)}$$

Using this estimated policy function, calculate the value function as

$$\begin{aligned} V(a_t, \varepsilon_{0t}, \varepsilon_{1t}) &= [1 - \widehat{p}(a_t)] \Big(\theta_1 a_t + \varepsilon_{0t} + \beta E[V(a_{t+1}, \varepsilon_{0t+1}, \varepsilon_{1t+1})|a_t, i_t = 0]\Big) \\ &\quad + \widehat{p}(a_t)\Big(R + \varepsilon_{1t} + \beta E[V(a_{t+1}, \varepsilon_{0t+1}, \varepsilon_{1t+1})|a_t, i_t = 1]\Big) \end{aligned}$$

Take expectation with respect to ε_t's, then

$$\begin{aligned} V(a_t) \equiv E[V(a_t, \varepsilon_{0t}, \varepsilon_{1t})] &= [1 - \widehat{p}(a_t)] \Big(\theta_1 a_t + E[\varepsilon_{0t}|a_t, i_t = 0] + \beta E[V(a_{t+1})|a_t, i_t = 0]\Big) \\ &\quad + \widehat{p}(a_t)\Big(R + E[\varepsilon_{1t}|a_t, i_t = 1] + \beta E[V(a_{t+1})|a_t, i_t = 1]\Big) \end{aligned}$$

Note also that

$$\begin{aligned} E[\varepsilon_{0t}|a_t, i_t = 0] &= 0.5772 - \log[1 - \widehat{p}(a_t)] \\ E[\varepsilon_{1t}|a_t, i_t = 1] &= 0.5772 - \log \widehat{p}(a_t) \end{aligned}$$

We have only 5 values of a_t, so the above equations can be rewritten using vectors. It is very easy to solve for V. Now use $V(a_t)$ to derive the policy function $\overline{p}(a_t)$, the probability that a firm replaces the machine when it has a_t.

$$\begin{aligned} \overline{p}(a_t) &= \Pr\Big(\theta_1 a_t + \varepsilon_{0t} + \beta E[V(a_{t+1})|a_t, i_t = 0] > R + \varepsilon_{1t} + \beta E[V(a_{t+1})|a_t, i_t = 1]\Big) \\ &= \frac{e^{R + \beta E[V(a_{t+1})|a_t, i_t = 1]}}{e^{\theta_1 a_t + \beta E[V(a_{t+1})|a_t, i_t = 0]} + e^{R + \beta E[V(a_{t+1})|a_t, i_t = 1]}} \end{aligned}$$

Use a moment condition, $E[i_j - \overline{p}(a_j)|a_j] = 0$ to do GMM estimation.

$$E[g_j(\theta)] \equiv E\begin{bmatrix} i_j - \overline{p}(a_j) \\ a_j[i_j - \overline{p}(a_j)] \end{bmatrix} = 0$$

Its variance would be

$$Var(\widehat{\theta}) = \frac{1}{N}\left(E\left[\frac{\partial g_j(\theta)}{\partial \theta'}\right]\right)^{-1} E\left[g_j(\theta) g_j(\theta)'\right] \left(E\left[\frac{\partial g_j(\theta)'}{\partial \theta}\right]\right)^{-1}$$

	θ_1	R
GMM estimate	-1.1491	-4.4553
(s.e.)	(0.0774)	(0.3317)