ECON 271C homework #1

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RED texts are comments by the professor.

Investment and adjustment costs: Lucas and Prescott

1. (a) 5/5 points

Write down planner's problem in terms of sequences of variables.

$$V(k_{0a}, k_{0b}) = \max_{\{k_{t+1,s}, i_{ts}, n_{ts}\}} \sum_{t=0}^{\infty} \beta^t u\left(\frac{1}{2}af(k_{ta}, n_{ta}) + \frac{1}{2}bf(k_{tb}, n_{tb}) - \frac{1}{2}g\left(\frac{i_{ta}}{k_{ta}}\right)k_{ta} - \frac{1}{2}g\left(\frac{i_{tb}}{k_{tb}}\right)k_{tb}\right)$$

Using the following resource contraints,

$$\begin{aligned}(1-\delta)k_{ts} + i_{ts} &= k_{t+1,s} \\ n_{ta} + n_{tb} &= 1\end{aligned}$$

we can write the Bellman equations as

$$\begin{aligned}V(k_a, k_b) = \max_{\{k'_a, k'_b, n_a\}} \quad &\Bigg\{u\Bigg(\frac{1}{2}af(k_a, n_a) + \frac{1}{2}bf(k_b, 1-n_a) \\ &- \frac{1}{2}g\left(\frac{k'_a - [1-\delta]k_a}{k_a}\right)k_a - \frac{1}{2}g\left(\frac{k'_b - [1-\delta]k_b}{k_b}\right)k_b\Bigg) + \beta V(k'_a, k'_b)\Bigg\}\end{aligned}$$

FOC's are

$$\begin{aligned}\frac{1}{2}u'(c)\ g'\left(\frac{k'_a - [1-\delta]k_a}{k_a}\right) &= \beta V_1(k'_a, k'_b) &(1)\\ \frac{1}{2}u'(c)\ g'\left(\frac{k'_b - [1-\delta]k_b}{k_b}\right) &= \beta V_2(k'_a, k'_b) &(2)\\ af_2(k_a, n_a) &= bf_2(k_b, 1-n_a) &(3)\end{aligned}$$

Envelope conditions are

$$\begin{aligned}V_1(k_a, k_b) &= u'(c)\left[\frac{1}{2}af_1(k_a, n_a) - \frac{1}{2}g\left(\frac{k'_a - [1-\delta]k_a}{k_a}\right) + \frac{1}{2}g'\left(\frac{k'_a - [1-\delta]k_a}{k_a}\right)\frac{k'_a}{k_a}\right] &(4)\\ V_2(k_a, k_b) &= u'(c)\left[\frac{1}{2}bf_1(k_b, 1-n_a) - \frac{1}{2}g\left(\frac{k'_b - [1-\delta]k_b}{k_b}\right) + \frac{1}{2}g'\left(\frac{k'_b - [1-\delta]k_b}{k_b}\right)\frac{k'_b}{k_b}\right] &(5)\end{aligned}$$

Combining (1) and (4), and (2) and (5), we obtain the following Euler equations.

$$\begin{aligned}u'(c)\ g'\left(\frac{k'_a - [1-\delta]k_a}{k_a}\right) &= \beta u'(c')\left[af_1(k'_a, n'_a) - g\left(\frac{k''_a - [1-\delta]k'_a}{k'_a}\right) + g'\left(\frac{k''_a - [1-\delta]k'_a}{k'_a}\right)\frac{k''_a}{k'_a}\right]\\ u'(c)\ g'\left(\frac{k'_b - [1-\delta]k_b}{k_b}\right) &= \beta u'(c')\left[bf_1(k'_b, 1-n'_a) - g\left(\frac{k''_b - [1-\delta]k'_b}{k'_b}\right) + g'\left(\frac{k''_b - [1-\delta]k'_b}{k'_b}\right)\frac{k''_b}{k'_b}\right]\end{aligned}$$

Suppose that in the steady state, neither type of firm has zero capital. In the steady state, $k_a = k'_a = \overline{k}_a$, $k_b = k'_b = \overline{k}_b$, $n_a = \overline{n}_a$, and $c = c' = \overline{c}$. So the Euler equations have the following form.

$$\begin{aligned} g'(\delta) &= \beta\left[af_1(\overline{k}_a, \overline{n}_a) - g(\delta) + g'(\delta)\right] \\ g'(\delta) &= \beta\left[bf_1(\overline{k}_b, 1-\overline{n}_a) - g(\delta) + g'(\delta)\right] \end{aligned}$$

Combining these two together,

$$af_1(\overline{k}_a, \overline{n}_a) = bf_1(\overline{k}_b, 1-\overline{n}_a)$$

Using the fact that f is CRS (and thus homogeneous of degree 1), f_1 is homogeneous of degree 0.

$$af_1\left(\frac{\overline{k}_a}{\overline{n}_a}, 1\right) = bf_1\left(\frac{\overline{k}_b}{1-\overline{n}_a}, 1\right)$$

Since f_1 is decreasing in its first argument and $a < b$,

$$\frac{\overline{k}_a}{\overline{n}_a} < \frac{\overline{k}_b}{1-\overline{n}_a}$$

Now evaluate (3) at the steady state and use the fact that f_2 is homegeneous of degree 0.

$$af_2\left(\frac{\overline{k}_a}{\overline{n}_a}, 1\right) = bf_2\left(\frac{\overline{k}_b}{1-\overline{n}_a}, 1\right)$$

Since f_2 is increasing in its first argument and $a < b$,

$$\frac{\overline{k}_a}{\overline{n}_a} > \frac{\overline{k}_b}{1-\overline{n}_a}$$

There is a contradiction. Now we have either $\overline{k}_a = 0$ or $\overline{k}_b = 0$. Obviously, the social planner would choose to utilize b type of firms only, since it is more productive. So $\overline{k}_a = 0$, or in other words, a type of firms will have zero capital in the steady state.

(b) 3/5

Both types of firms start with k_0 amount of captial. a type of firms would wait without investing anything until k_a depreciates completely. (Not necessarily. Not if total k is very low.) b type of firms would invest so that their capital level converges to one in the steady state. These are characterized by the following equations.

$$\begin{aligned} af_2(k_{ta}, n_{ta}) &= bf_2(k_{tb}, 1-n_{ta}) \\ k_{t+1,a} &= (1-\delta)k_{ta} \\ u'(c_t)\ g'\left(\frac{k_{t+1,b} - [1-\delta]k_{tb}}{k_{tb}}\right) &= \beta u'(c_{t+1})\Bigg[bf_1(k_{t+1,b}, 1-n_{t+1,a}) \\ &\quad -g\left(\frac{k_{t+2,b} - [1-\delta]k_{t+1,b}}{k_{t+1,b}}\right) + g'\left(\frac{k_{t+2,b} - [1-\delta]k_{t+1,b}}{k_{t+1,b}}\right)\frac{k_{t+2,b}}{k_{t+1,b}}\Bigg] \end{aligned}$$

2. **15/15** **(a)**

The firm would choose $q_t = k_t$ for any realizations of marginal cost since it is always less than price. Write down planner's problem (firm's problem) in terms of sequences of variables.

$$V(k_0, c_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t E\left[(p - c_t)k_t - \frac{a(k_{t+1} - k_t)^2}{2} \middle| c_0\right]$$

The dynamic programming would be

$$V(k, c) = \max_{k'} \left\{(p - c)k - \frac{a(k' - k)^2}{2} + \beta E[V(k', c')|c]\right\}$$

where expectation is with respect to c' whose distribution is given as

$$c' = \rho c + \varepsilon, \quad \varepsilon \text{ is i.i.d. with mean } \bar{\varepsilon} > 0$$

(b)

FOC is

$$a(k' - k) = \beta E[V_1(k', c')|c]$$

The envelope condition is

$$V_1(k, c) = p - c + a(k' - k)$$

Combining these two, we obtain the following Euler equation.

$$a(k' - k) = \beta E[(p - c') + a(k'' - k')|c]$$

Using the law of motion and the moment condition, we can rewrite this in terms of sequences of variables as

$$a(k_{t+1} - k_t) = \beta(p - \rho c_t - \bar{\varepsilon}) + \beta a E[k_{t+2} - k_{t+1}|c_t]$$

(c)

Using the guess function, this writes as

$$a(A_0 + A_1 p + A_2 c_t) = \beta(p - \rho c_t - \bar{\varepsilon}) + \beta a E[A_0 + A_1 p + A_2 c_{t+1}|c_t]$$

or

$$a(A_0 + A_1 p + A_2 c_t) = \beta(p - \rho c_t - \bar{\varepsilon}) + \beta a(A_0 + A_1 p + A_2 \rho c_t + A_2 \bar{\varepsilon})$$

Using the method of undertermined coefficients, we have

$$\begin{aligned} A_2 &= -\frac{\beta\rho}{a(1 - \beta\rho)} \\ A_1 &= \frac{\beta}{a(1 - \beta)} \\ A_0 &= -\frac{\beta\bar{\varepsilon}}{a(1 - \beta)(1 - \beta\rho)} \end{aligned}$$

Note that we do not have variation of p in this model, but that the result would be the same even if we guess as $k_{t+1} - k_t = B_0 + B_2 c_t$, in the sense that $B_2 = A_2$ and $B_0 = A_0 + A_1 p$.

(d)

The solution can be rewritten as

$$a(k_{t+1} - k_t) = \frac{\beta}{1-\beta} p - \frac{\beta\rho}{1-\beta\rho} c_t - \frac{\beta\bar{\varepsilon}}{(1-\beta)(1-\beta\rho)}$$

or

$$a(k_{t+1} - k_t) = (\beta + \beta^2 + \cdots)p - (\beta\rho + \beta^2\rho^2 + \cdots)c_t - (\beta + \beta^2 + \cdots)(1 + \beta\rho + \cdots)\bar{\varepsilon}$$

The term in the left hand side stands for the marginal cost of new investment. The term in the right hand side is exactly its expected marginal revenue. One unit of new investment would yield a revenue $\beta(p - c_{t+1})$ in the next period. Its expected value is $\beta(p - \rho c_t - \bar{\varepsilon})$. It yields a revenue $\beta^2(p - c_{t+2})$ in the two periods later. Its expected value is $\beta^2(p - \rho^2 c_t - \rho\bar{\varepsilon} - \bar{\varepsilon})$. An infinite sum of the expected value of revenue in next periods is the term in the RHS. So the solution is nothing but the marginal cost of investment being equal to its expected marginal revenue.

(e)

Yes, they are autocorrelated. Recall from the Euler equation that

$$\beta a E[k_{t+2} - k_{t+1} | c_t] = a(k_{t+1} - k_t) - \beta(p - \rho c_t - \bar{\varepsilon})$$

or

$$E[k_{t+2} - k_{t+1} | c_t] = \frac{1}{\beta}(k_{t+1} - k_t) - \frac{1}{a}(p - \rho c_t - \bar{\varepsilon})$$

So they are autocorrelated. We can write AR1 model as

$$k_{t+2} - k_{t+1} = \frac{1}{\beta}(k_{t+1} - k_t) - \frac{1}{a}(p - \rho c_t - \bar{\varepsilon}) + u_t$$

(f)

Firms which get small marginal cost shock will grow fast, while other firms will grow slow. As time goes by, there is a widely spread distribution of history of shock firms get. Some firm would encounter repeatedly low MC to grow very large, but other firms do not. Price would go down as firms grow their size, and then some firms would not produce anything because its marginal cost exceeds price. In the end, "a small number of large firms would produce almost all of the demand" (?), while another small number of small firms are luckily provided with very good technique (low MC) to produce the rest few of the demand.

Span of Control

4. 15/15 (a)

Assume that good 1 is a numeraire. Firm 1 solves

$$\pi_1(z) = \max_n z n^\alpha - wn$$

FOC reads as

$$\alpha z n^{\alpha-1} = w$$

So,

$$n_1(z) = \left(\frac{\alpha z}{w}\right)^{\frac{1}{1-\alpha}} \tag{6}$$

Firm 2 solves

$$\pi_2(z) = \max_n p\theta z n^\alpha - wn$$

where p is good 2's price. FOC reads as

$$\alpha p\theta z n^{\alpha-1} = w$$

So,

$$n_2(z) = \left(\frac{\alpha p\theta z}{w}\right)^{\frac{1}{1-\alpha}} \tag{7}$$

Find z^0 such that $\pi_1(z^0) = \pi_2(z^0)$, or

$$z\left(\frac{\alpha z}{w}\right)^{\frac{\alpha}{1-\alpha}} - w\left(\frac{\alpha z}{w}\right)^{\frac{1}{1-\alpha}} = p\theta z\left(\frac{\alpha p\theta z}{w}\right)^{\frac{\alpha}{1-\alpha}} - w\left(\frac{\alpha p\theta z}{w}\right)^{\frac{1}{1-\alpha}}$$

Arranging terms,

$$\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)\left(\frac{z}{w^\alpha}\right)^{\frac{1}{1-\alpha}} = \alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)\left(\frac{p\theta z}{w^\alpha}\right)^{\frac{1}{1-\alpha}}$$

It is clear that as long as $p\theta \neq 1$, there is no such z^0. Since the utility function is $u(c_1, c_2) = \log c_1 + \log c_2$, so both goods should be produced. Therefore $p\theta = 1$, or

$$p = \frac{1}{\theta} \tag{8}$$

so that all managers are indifferent between producing good 1 and good 2. As a consequence, $n_1(z) = n_2(z) \equiv n(z)$. Now we have to find who operates firms and who works. There is a cutoff z^* such that people with $z > z^*$ operate firms, and those with $z < z^*$ work. It is determined by the following equation.

$$z^* n(z^*)^\alpha - wn(z^*) = w \tag{9}$$

Also, it has to satisfy feasibility.

$$1 - F(z^*) + \int_{z^*} n(z)dF(z) = 1 \tag{10}$$

We still need to determine how many people operate firm 1. With p given in (8), the representative consumer maximizes a utility

$$\max \log c_1 + \log c_2 \qquad \text{subject to } c_1 + pc_2 = I$$

where $I = w$ if a worker, and $I = \pi(z)$ if a manager. As a result,

$$c_1 = \frac{I}{2}, \quad c_2 = \frac{I}{2p} \tag{11}$$

Recall (8), so c_2 has to be produced θ times as much as c_1 in the equilibrium. This and equations (6) to (11) characterize the competitive equilibrium. In one of the equilibria, half of people with z manage firm 1, and the other half with the same z manage firm 2. This is because c_2 is θ times as productive as c_1. There could be more equilibria in which other set of people manage firm 1. Also, since $n_1(z) = n_2(z)$, the relative size of firms between the two sectors would be the same for the same z. Actually it depends on manager's talent.

(b)

Now firm 1 sovles

$$\pi_1(z) = \max_n zn^\alpha - w(n + n_0)$$

so

$$n_1(z) = \left(\frac{\alpha z}{w}\right)^{\frac{1}{1-\alpha}} \tag{12}$$

Firm 2 solves

$$\pi_2(z) = \max_n pzn^\alpha - wn$$

so

$$n_2(z) = \left(\frac{\alpha pz}{w}\right)^{\frac{1}{1-\alpha}} \tag{13}$$

Find z^0 which satisfies $\pi_1(z^0) = \pi_2(z^0)$, or

$$z\left(\frac{\alpha z}{w}\right)^{\frac{\alpha}{1-\alpha}} - w\left[\left(\frac{\alpha z}{w}\right)^{\frac{1}{1-\alpha}} + n_0\right] = pz\left(\frac{\alpha pz}{w}\right)^{\frac{\alpha}{1-\alpha}} - w\left(\frac{\alpha pz}{w}\right)^{\frac{1}{1-\alpha}}$$

Arranging terms,

$$\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)\left(\frac{z^0}{w^\alpha}\right)^{\frac{1}{1-\alpha}} - wn_0 = \alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)\left(\frac{pz^0}{w^\alpha}\right)^{\frac{1}{1-\alpha}} \tag{14}$$

We can infer that p would be less than 1, since otherwise the RHS is always greater than the LHS for every z^0, and thus there's no such z^0. Also, since $p < 1$, for $z < z^0$, it would be that $\pi_1(z) < \pi_2(z)$ due to large fixed cost wn_0, while $\pi_1(z) > \pi_2(z)$ for $z > z^0$. In other words, if $z < z^0$, she will manage firm 2, and if $z > z^0$, she will manage firm 1. Now find z^* with which a person is indifferent between managing firm 2 and working.

$$z^* n_2(z^*)^\alpha - wn_2(z^*) = w \tag{15}$$

For feasibility,

$$1 - F(z^*) + \int_{z^*}^{z^0} n_2(z)dF(z) + \int_{z^0} n_1(z)dF(z) = 1 \tag{16}$$

People consume according to (11). So c_1 has to be produced p times as much as c_2. This implies the following condition.

$$p\int_{z^*}^{z^0} zn_2(z)^\alpha dF(z) = \int_{z^0} zn_1(z)^\alpha dF(z) \tag{17}$$

Equations (11) to (17) define the competitive equilbrium. We have 8 unknowns $n_1, n_2, c_1, c_2, z^0, z^*, p$ and w and 8 equations, so we can solve for them. Now we would have a different prediction on the relative size of firms. All the managers of firm 1 have higher z than those of firm 2 and $p < 1$. So from (12) and (13), any firm 1 has bigger size than any firm 2.

(c)

Firm 1 solves

$$\pi_1(z) = \max_n zn^\alpha - wn$$

so

$$n_1(z) = \left(\frac{\alpha z}{w}\right)^{\frac{1}{1-\alpha}} \tag{18}$$

Firm 2 solves

$$\pi_2(z) = \max_n pg(z)n^\alpha - wn$$

so

$$n_2(z) = \left[\frac{\alpha pg(z)}{w}\right]^{\frac{1}{1-\alpha}} \tag{19}$$

Find z^0 such that $\pi_1(z^0) = \pi_2(z^0)$, or

$$z\left(\frac{\alpha z}{w}\right)^{\frac{\alpha}{1-\alpha}} - w\left(\frac{\alpha z}{w}\right)^{\frac{1}{1-\alpha}} = pz\left[\frac{\alpha pg(z)}{w}\right]^{\frac{\alpha}{1-\alpha}} - w\left[\frac{\alpha pg(z)}{w}\right]^{\frac{1}{1-\alpha}}$$

Arranging terms,

$$\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)\left(\frac{z}{w^\alpha}\right)^{\frac{1}{1-\alpha}} = \alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)\left[\frac{pg(z)}{w^\alpha}\right]^{\frac{1}{1-\alpha}}$$

z^0 satisfies

$$z^0 = pg(z^0) \tag{20}$$

For appropriate p, there exists z^0 satisfying the above equation. Since $g(z)$ is strictly increasing and concave, for $z < z^0$, it would be that $\pi_1(z) < \pi_2(z)$, while $\pi_1(z) > \pi_2(z)$ for $z > z^0$. In other words, if $z < z^0$, she will manage firm 2, and if $z > z^0$, she will manage firm 1. Based on this, the rest of conditions would have the same form with (15), (16), and (17). Equations (15) to (20) together with (11) define the competitive equilbrium. We have 8 unknowns $n_1, n_2, c_1, c_2, z^0, z^*, p$ and w and 8 equations, so we can solve for them.

5. I don't know how to solve this problem.