

ECON 271C homework #2
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Industry equilibrium

7. Industry equilibrium (a)

I assume that a private firm becomes public as soon as it pays c_0 . Denote private firm's current period profit and present value by π^{PR} and V^{PR} , respectively. π and V without any superscript stand for public firm's profit function and value function, respectively. Private firms have a capacity constraint, while public firms do not, so

$$\pi^{\text{PR}}(\varphi, p) \leq \pi(\varphi, p) \quad \text{for any } \varphi \text{ and } p$$

with equality when a capacity constraint is slack. Then, value functions would satisfy

$$V(\varphi, p) = \max \left\{ 0, \pi(\varphi, p) + \delta \int V(\varphi', p) F(d\varphi'|\varphi) \right\}$$

$$V^{\text{PR}}(\varphi, p) = \max \left\{ 0, \pi^{\text{PR}}(\varphi, p) + \delta \int V^{\text{PR}}(\varphi', p) F(d\varphi'|\varphi), \pi(\varphi, p) - c_0 + \delta \int V(\varphi', p) F(d\varphi'|\varphi) \right\}$$

Also, there is a value function for an entrant. I assume that a firm behaves as a private firm the period when it enters.

$$V^e(p) = \int V^{\text{PR}}(\varphi, p) G(d\varphi) - c_e$$

For an outside firm to be indifferent between entering and staying out, it should be that $V^e(p^*) = 0$, or equivalently,

$$\int V^{\text{PR}}(\varphi, p^*) G(d\varphi) = c_e \tag{1}$$

Suppose that there is no ex post uncertainty, in the sense that an existing firm has the same φ forever. Then, the value functions would be

$$V(\varphi, p) = \max \left\{ 0, \frac{\pi(\varphi, p)}{1 - \delta} \right\}$$

$$V^{\text{PR}}(\varphi, p) = \max \left\{ 0, \frac{\pi^{\text{PR}}(\varphi, p)}{1 - \delta}, \frac{\pi(\varphi, p)}{1 - \delta} - c_0 \right\}$$

In this case, there exist two thresholds φ^* and φ^{**} such that a private firm who gets φ decides to

| | |
|-----------------------------|--|
| exit | if $\varphi < \varphi^*$, |
| stay as private | if $\varphi^* \leq \varphi < \varphi^{**}$, and |
| pay c_0 and become public | if $\varphi \geq \varphi^{**}$ |

These thresholds satisfy

$$\frac{\pi^{\text{PR}}(\varphi^*, p^*)}{1 - \delta} = 0$$

$$\frac{\pi^{\text{PR}}(\varphi^{**}, p^*)}{1 - \delta} = \frac{\pi(\varphi^{**}, p^*)}{1 - \delta} - c_0$$

A stationary equilibrium would be $(p^*, \varphi^*, \varphi^{**}, M^*)$ that satisfies (1), the above two equations, and the following condition of an invariant measure of firms.

$$D \left(\int_{\varphi^*}^{\varphi^{**}} x^{\text{PR}}(\varphi, p^*) M^* G(d\varphi) + \int_{\varphi^{**}} x(\varphi, p^*) M^* G(d\varphi) \right) = p^*$$

Now consider the case where φ changes over time. In this case, a public firm can exit if it gets a very bad shock. So there are three thresholds φ^* , φ^{**} and φ^{***} . The first two are defined as before and φ^{***} is such that a public firm exits when it gets $\varphi < \varphi^{***}$. These lead to the following equations.

$$\pi^{\text{PR}}(\varphi^*, p^*) + \delta \int V^{\text{PR}}(\varphi', p^*) F(d\varphi' | \varphi^*) = 0 \quad (2)$$

$$\pi^{\text{PR}}(\varphi^{**}, p^*) + \delta \int V^{\text{PR}}(\varphi', p^*) F(d\varphi' | \varphi^{**}) = \pi(\varphi^{**}, p^*) - c_0 + \delta \int V(\varphi', p^*) F(d\varphi' | \varphi^{**}) \quad (3)$$

$$\pi(\varphi^{***}, p^*) + \delta \int V(\varphi', p^*) F(d\varphi' | \varphi^{***}) = 0 \quad (4)$$

From these conditions, we can see that $\varphi^{***} \leq \varphi^* \leq \varphi^{**}$. Also there are two measures, one for private firms and the other for public firms. Denote this as μ^{PR} and μ , respectively. In a stationary equilibrium, these should be invariant, and thus satisfy

$$\mu^{\text{PR}}([\varphi^*, s]) = \lambda^* [G(s) - G(\varphi^*)] + \int [F(s|\varphi) - F(\varphi^*|\varphi)] \mu^{\text{PR}}(d\varphi) \quad \text{for } s \leq \varphi^{**} \quad (5)$$

$$\begin{aligned} \mu([\varphi^{***}, s]) &= \lambda^* \max \{0, G(s) - G(\varphi^{**})\} + \int \max \{0, F(s|\varphi) - F(\varphi^{**}|\varphi)\} \mu^{\text{PR}}(d\varphi) \\ &\quad + \int [F(s|\varphi) - F(\varphi^{***}|\varphi)] \mu(d\varphi) \end{aligned} \quad (6)$$

Under the above measures, p^* should support total demand.

$$D \left(\int_{\varphi^*}^{\varphi^{**}} x^{\text{PR}}(\varphi, p^*) \mu^{\text{PR}}(d\varphi) + \int_{\varphi^{***}} x(\varphi, p^*) \mu(d\varphi) \right) = p^* \quad (7)$$

A stationary equilibrium is $(V^{\text{PR}}, V, p^*, \varphi^*, \varphi^{**}, \varphi^{***}, \mu^{\text{PR}}, \mu, \lambda^*)$ that satisfies equations (1) to (7) as well as the definition of value functions.

(b)

It is clear that value functions V^{PR} and V are uniquely defined under some assumptions on π^{PR} , π and F . Also φ^* , φ^{**} , φ^{***} and p^* are uniquely defined by equations (1) to (4) given V^{PR} and V . Equations (5) to (7) characterize unique λ^* , and thus μ^{PR} and μ as well. Therefore, there exists a unique stationary equilibrium if μ^{PR} and μ are uniquely defined by λ^* . Temporarily suppose that the support of measures are discrete, and transform (5) and (6) into equations in vector version of measures.

$$\begin{aligned} \mu^{\text{PR}} &= \lambda^* \nu^{\text{PR}} + T_1 \mu^{\text{PR}} \\ \mu &= \lambda^* \nu + S \mu^{\text{PR}} + T_2 \mu \end{aligned}$$

This leads to the following solution.

$$\begin{aligned}\mu^{\text{PR}} &= (I - T_1)^{-1} \lambda^* \nu^{\text{PR}} = \lambda^* \sum_{n=0}^{\infty} T_1^n \nu^{\text{PR}} \\ \mu &= (I - T_2)^{-1} (\lambda^* \nu + S \mu^{\text{PR}}) = \sum_{n=0}^{\infty} T_2^n (\lambda^* \nu + S \mu^{\text{PR}})\end{aligned}$$

Analogously the measure functions would satisfy

$$\begin{aligned}\mu^{\text{PR}} &= \lambda^* \sum_{t=0}^{\infty} \alpha_{1t} \tilde{\mu}_{1t} \\ \mu &= \lambda^* \sum_{t=0}^{\infty} \alpha_{2t} \tilde{\mu}_{2t} + \sum_{t=0}^{\infty} \alpha_{3t} \tilde{\mu}_{3t}\end{aligned}$$

where $\tilde{\mu}_{1t}$ is a conditional cdf of private firms' shock after t periods of survival, $\tilde{\mu}_{2t}$ is a conditional cdf of public firms' shock after t periods of survival, $\tilde{\mu}_{3t}$ is a conditional cdf of shock of private firms becoming public after t periods of survival, and α_{1t}, α_{2t} and α_{3t} are corresponding survival probabilities. These measure functions exist and are unique if

$$\sum_{t=0}^{\infty} \alpha_{1t} < \infty, \quad \sum_{t=0}^{\infty} \alpha_{2t} < \infty \quad \text{and} \quad \sum_{t=0}^{\infty} \alpha_{3t} < \infty$$

(c)

As shown above, φ^{***} is smaller than φ^* . So the interval in which private firms exist is narrower than and is included in the interval where public firms exist. Also the average of the profit made by public firms is higher than that made by private firms. This is because public firms have a positive probability that they get a very high shock in the future. In the stationary equilibrium, it should be the case that the change rate of private firms into public firms is the same with the exit rate of public firms, and also that the change rate plus the exit rate of private firms is the same with the entry rate.

(d)

If there is no option to become public, there exist only one type of firms. Then the stationary equilibrium has the same properties with that in the original problem where there are only firms without capacity constraint. The only difference between the equilibria in the two models is profit function of firms according to capacity constraint.

11. Durable goods (a)

Let M' be the number of firms after entry. It is trivial that $M' \leq 1$ as long as $c_e > 0$. M' units of durable goods will be produced since there is no production cost. New durable goods would be

consumed by those whose valuation θ is high. Then, the social benefit without considering entry cost is

$$\begin{aligned} R(M', s) &= \int_{1-M'}^1 \theta z d\theta + \int_{1-M'-s}^{1-M'} \theta \alpha z d\theta \\ &= \frac{1}{2} z (2M' - M'^2) + \frac{1}{2} \alpha z [2s(1 - M') - s^2] \end{aligned}$$

if $M' + s \leq 1$, and

$$\begin{aligned} R(M', s) &= \int_{1-M'}^1 \theta z d\theta + \int_0^{1-M'} \theta \alpha z d\theta \\ &= \frac{1}{2} z (2M' - M'^2) + \frac{1}{2} \alpha z (1 - M')^2 \end{aligned}$$

if $M' + s \geq 1$. Given this, the value function of social planner's problem is

$$V(M, s) = \max_e \left\{ R(M + e, s) - ec_e + \delta V\left([1 - \lambda][M + e], M + e\right) \right\}$$

(b)

FOC writes as

$$R_1(M + e^*, s) - c_e + (1 - \lambda)\delta V_1\left([1 - \lambda][M + e^*], M + e^*\right) + \delta V_2\left([1 - \lambda][M + e^*], M + e^*\right) = 0$$

where $e^* = e(M, s)$ denotes optimal entry. Envelope theorem yields

$$V_1(M, s) = R_1(M + e^*, s) + (1 - \lambda)\delta V_1\left([1 - \lambda][M + e^*], M + e^*\right) + \delta V_2\left([1 - \lambda][M + e^*], M + e^*\right)$$

$$V_2(M, s) = R_2(M + e^*, s)$$

Recall from the definition of R that when $M' + s \leq 1$,

$$R_1(M', s) = z(1 - M' - \alpha s) \geq 0$$

$$R_2(M', s) = \alpha z(1 - M' - s) \geq 0$$

and also when $M' + s \geq 1$,

$$R_1(M', s) = (1 - \alpha)z(1 - M') \geq 0$$

$$R_2(M', s) = 0$$

So in any case,

$$R_{11} < 0, R_{12} \leq 0 \text{ and } R_{22} \leq 0$$

Using FOC, the first envelope condition is equivalent to

$$V_1(M, s) = c_e$$

So FOC can be rewritten as

$$R_1(M + e^*, s) + \delta V_2\left([1 - \lambda][M + e^*], M + e^*\right) = [1 - (1 - \lambda)\delta]c_e \quad (8)$$

Also differentiating $V_1(M, s)$ with respect to s yields

$$V_{12}(M, s) = 0$$

We want to show that $e(M, s)$ is decreasing in both M and s . First, differentiate $V_2(M, s)$ with respect to M .

$$V_{12}(M, s) = (1 + e_1)R_{12}(M + e^*, s)$$

For this to be 0, it should be that $1 + e_1 = 0$, and thus $e_1 < 0$. Now suppose that $e_2 > 0$. We have $V_{22} = R_{22} \leq 0$. Let s increase with M being fixed. Then, the first term in LHS of (8) strictly decreases, since e^* increases and $R_{11} < 0$. The second term weakly decreases, but RHS does not change, which is a contradiction. So $e(M, s)$ is decreasing in both M and s .

This property of policy function implies that the number of firms is mean reverting. In other words, if there are too many firms, entry will be less and if there are too few firms, entry will be large. Accordingly the sales of the new durable good would converge to some number.

(c)

Since those who have high valuation θ consume new durable good, all consumers with valuation higher than $1 - M - e$ would prefer new, but others would prefer used. So one with valuation $1 - M - e$ is indifferent between two. Denote prices of new good and used good by p_n and p_u , respectively.

$$(1 - M - e)z - p_n + \delta p'_u = (1 - M - e)\alpha z - p_u$$

Also, one with valuation $1 - M - e - s$ should be indifferent between buying used and buying nothing.

$$(1 - M - e - s)\alpha z - p_u = 0$$

These yield

$$\begin{aligned} p_u &= (1 - M - e - s)\alpha z \\ p_n &= (1 - M - e)z - (1 - M - e)\alpha z + p_u + \delta p'_u \\ &= (1 - M - e)z - s\alpha z + \delta(1 - M' - e' - s')\alpha z \end{aligned}$$

(d)

When there is a production cost varying with time spent in the industry, all of the producing firms have low cost and get positive profit. Since λ portion of existing firms exit, in the stationary equilibrium, the same amount of new firms would enter with high cost of production. These firms may not produce in the first several periods, but after that, they would produce a good if they still survive.

12. Learning by doing (a)

Note that firm's profit is always greater than 0 due to the given form of cost function. So we do not have to consider the outside option if $V_0 = 0$. A firm with s has the following dynamic programming problem.

$$V^p(s) = \max_q \left\{ pq - \frac{(q - \alpha s)^2}{2} + \beta V([1 - \delta]s + \lambda q) \right\}$$

This can be written in the following form as well.

$$V^p(s) = \max_{s'} \left\{ p \cdot \frac{s' - (1 - \delta)s}{\lambda} - \frac{[s' - (1 - \delta + \alpha\lambda)s]^2}{2\lambda^2} + \beta V(s') \right\}$$

A firm considering whether to enter or not has $s = 0$. This would have a linear policy function, because a return function is quadratic in the state variable s . Guess $V(s) = A_0 + A_1 s + A_2 s^2$, and use the method of undertermined coefficient, and then we could get the exact value function in quadratic form, and a linear policy function.

(b)

A firm with s has the following problem.

$$V(s) = \max \left\{ V_0, \max_q \left\{ pq - \frac{(q - \alpha s)^2}{2} + \beta V([1 - \delta]s + \lambda q) \right\} \right\} = \max\{V_0, V^p(s)\} \quad (9)$$

Given that a firm decides to stay in the industry, the optimal output does not change. However, some firms would stop producing and exit because of high outside value. This is not likely to occur for the firm with high s , since it has a cost low enough to get a high profit from producing. In other words, firms would exit mostly in their early age, and as they spend more time in the industry, the probability that they exit decreases.

(c)

Let $\mu(s)$ be a measure of firms' experience. Note that given p , μ does not affect individual firm's decision on production and exit since q depends only on p and s , and exit decision depends only on V_0 and $V^p(s)$. As seen in (9), a firm with s stays in the industry with probability

$$\Pr(\text{stay}|s) = \Pr(V_0 < V^p(s))$$

An industry equilibrium is, therefore, p^* that satisfies

$$D \left(\int_0 q(p^*, s) \mu(ds) \right) = p^* \quad (10)$$

where $q(p, s)$ is an output policy function defined as

$$q(p, s) = \begin{cases} 0 & \text{with probability } 1 - \Pr(\text{stay}|s) \\ a_0 + a_1 p + bs \text{ solving } V^p(s) & \text{with probability } \Pr(\text{stay}|s) \end{cases}$$

We can see that under appropriate assumption there is a unique industry equilibrium p^* solving (10). This assumption includes that D is strictly decreasing. Now consider a stationary equilibrium. For an entrant to be indifferent between entering and staying out,

$$c_e = V^p(0) \Pr(\text{stay}|0) \quad (11)$$

Note that the above equation defines p^* . For μ to be invariant, it should be satisfied that

$$\mu(s') = \mu(s) \Pr(\text{stay}|s) \quad \text{where } s' = (1 - \delta)s + \lambda(a + bs) \quad (12)$$

recursively from $s = 0$, and $\mu(s) = 0$ for any other s . A stationary equilibrium is (p^*, μ) that satisfies equations (10), (11), and (12).

There is a unique stationary equilibrium. First, it is clear that there is only one p^* solving (11) since $V^p(0)$ is strictly increasing in p . From (12), the whole μ function can be expressed in terms of $\mu(0)$. Note that $\Pr(\text{stay}|s) < 1$ for any s , since V_0 follows a normal distribution. So the number of firms with high s is less than that of firms with low s . Note also that s is bounded above. Therefore, there exists a unique $\mu(0)$ satisfying (10).

(d)

As \bar{x} increases, a higher outside value is likely to be drawn. For enough number of firms to stay in the industry, price has to be high enough. Also since the probability that a firm exits becomes bigger, the age of firms in the industry would be such that there are more young firms and less old firms. To see this with equations, suppose that p^* is fixed. For any value of s , more firms would exit than before. In other words, $\Pr(\text{stay}|s)$ decreases by far, especially for lower s . So for (11) to hold, p^* has to increase. This makes $V^p(s)$ increase for any value of s , but especially for higher s . Therefore, young firms exit with much larger probability, but old firms with a little bigger probability than before. This implies that the number of young firms should be very large for μ to be invariant. The total number of firms is ambiguous.

(e)

In the stationary equilibrium, a measure of firms' experience μ predicts that a small number of large firms exist, while there are many small firms. Also since $V^p(s)$ is an increasing function of s , large firms rarely exit, but small firms exit with high probability, which makes sense. But this result crucially depends on the assumption that V_0 is identically distributed for all firms. It is not reasonable that all firms have the same distribution of outside value. It is likely that a large firm has a bigger outside value than a small firm in the real world.

14. (a)

I assume that there is no cost of production. The value function is

$$V(s) = \max \left\{ O, ps + \delta \left[(1 - \lambda)V(s) + \lambda \int_0^1 V(z)G(dz) \right] \right\}$$

It is easy to incorporate production cost into the value function. Now the value of entry should be 0, so we have

$$V^e = \int_0^1 V(s)G(ds) - c_e = 0$$

Here I implicitly assume that an outside firm does not have an outside option O . Plug this back into the value function, then

$$V(s) = \max \left\{ O, ps + \delta [(1 - \lambda)V(s) + \lambda c_e] \right\}$$

So when $V(s) > O$,

$$V(s) = ps + \delta [(1 - \lambda)V(s) + \lambda c_e]$$

or equivalently,

$$V(s) = \frac{ps + \delta \lambda c_e}{1 - \delta(1 - \lambda)}$$

Therefore we can write

$$V(s) = \max \left\{ O, \frac{ps + \delta \lambda c_e}{1 - \delta(1 - \lambda)} \right\}$$

(b)

A firm exits if and only if

$$O > \frac{ps + \delta \lambda c_e}{1 - \delta(1 - \lambda)}$$

This condition is equivalent to

$$s < \frac{1}{p} \left(O[1 - \delta(1 - \lambda)] - \delta \lambda c_e \right) \equiv s^* \quad (13)$$

s^* denotes a threshold under which a firm exits. Note here that in some cases s^* can be less than 0, so the actual threshold would be 0, and also that in other cases s^* might be greater than 1, so the actual threshold becomes 1.

(c)

Differentiating s^* with respect to λ and δ , respectively, we get

$$\begin{aligned} \frac{ds^*}{d\lambda} &= \frac{\delta}{p} (O - c_e) \leq 0 \\ \frac{ds^*}{d\delta} &= \frac{1}{p} (-O[1 - \lambda] - \lambda c_e) \leq 0 \end{aligned}$$

The first result comes from the following inequality. Since $V(s) \geq O$ by definition,

$$c_e = \int_0^1 V(s)G(ds) \geq \int_0^1 OG(dz) = O$$

The second result is true when $O \geq 0$ in which case $c_e \geq O \geq 0$ holds as well. Note that if $O < 0$, we have $s^* < 0$, in which case the actual threshold is 0 instead of s^* defined above. This implies that if $O < 0$, the actual threshold does not change when δ changes slightly.

(d)

Let μ^* be the stationary measure of firms' shock. We have

$$D \left(\int_{s^*} s \mu^*(ds) \right) = p^* \quad (14)$$

and

$$\mu^*([s^*, s]) = M \left[G(s) - G(s^*) \right] + (1 - \lambda) \mu^*([s^*, s]) + \lambda \mu^*([s^*, 1]) \left[G(s) - G(s^*) \right]$$

where the first term is entry from outside, the second is those who were in the industry and get the same shock, and the last is those who were in the industry and get new shock which falls again within the original interval. Arrange terms to get

$$\lambda \mu^*([s^*, s]) = \left[M + \lambda \mu^*([s^*, 1]) \right] \left[G(s) - G(s^*) \right]$$

Evaluating the equation at $s = 1$,

$$\lambda \mu^*([s^*, 1]) = \left[M + \lambda \mu^*([s^*, 1]) \right] [1 - G(s^*)]$$

or equivalently,

$$\mu^*([s^*, 1]) = \frac{M[1 - G(s^*)]}{\lambda G(s^*)}$$

Plug this back into the original equation, then

$$\lambda \mu^*([s^*, s]) = \left[M + \frac{M[1 - G(s^*)]}{G(s^*)} \right] \left[G(s) - G(s^*) \right]$$

Arranging terms, we get

$$\mu^*([s^*, s]) = \frac{M \left[G(s) - G(s^*) \right]}{\lambda G(s^*)} \quad (15)$$

Now let c_e increase. Suppose that s^* increases as a result. From (15), μ^* weakly decreases at every point. Consider (14). Since there are weakly less total output produced by all firms, p^* weakly increases. This contradicts to (13), since when both p^* and c_e increase, s^* cannot increase. Therefore, when c_e increases, s^* should decrease.

(e)

When λ increases, both s^* and p^* decrease. We can prove this by way of contradiction. Let λ increase. First suppose that s^* increases. By the same argument as in (d), μ^* weakly decreases at every point, and p^* weakly increases due to fall in total output. This leads to a contradiction since when both λ and p^* increase, s^* cannot increase from (13). To see this, recall from (c) that for fixed p , an increase in λ leads to a decrease in s^* , and that an increase in p^* leads to a further decrease in s^* .

Therefore, s^* decreases. Again by the same argument as in (d), μ^* weakly increases at every point, and p^* weakly decreases due to rise in total output, which completes the proof for the case when λ

increases. The same logic applies to the case when δ increases. So when δ increases, both s^* and p^* decrease.

(f)

In this model, c_e , λ , and δ affect survival of a firm through s^* . The result is intuitive. When cost of entry increases, it is likely that less firms enter, inducing price to increase. So a firm with low s can still get a positive expected value. If λ is big, a firm gets a new shock with large probability, so a firm with low s would expect to get a good shock in the future and thus want to stay. If δ is large, a future profit has a large present value, so a firm with low s has a value big enough to stay. But no firm grows endogenously, since shock is exogenous and lasts for long periods when λ is low. So this model does not give any prediction on growth of firm.