ECON 271C homework #3

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Industry equilibrium: General equilibrium

15. (a)

To avoid confusion, let $g(n)$ be a production function and f be a fixed cost. Firm's profit function is defined by

$$\pi(z,w) = \max_n zg(n) - w(n+f)$$

FOC yields

$$zg'(n) = w$$

or equivalently,

$$g'(n) = \frac{w}{z}$$

Since g is strictly concave, optimal n is decreasing in $\frac{w}{z}$. Using this, write the value function of the firm as

$$V(z,w) = \max\left\{0\ ,\ \pi(z,w) + \beta \int V(z',w)F(dz'|z)\right\}$$

where $F(z'|z)$ is a first order Markov process. A firm exits if it gets $z < z^*$ where z^* is a threshold satisfying

$$\pi(z^*,w) + \beta \int V(z',w)F(dz'|z^*) = 0 \tag{1}$$

An entrant has a value

$$V^e(w) = \int V(z,w)G(dz) - c_e w$$

So at the equilibrium, it should be satisfied that

$$\int V(z,w)G(dz) = c_e w \tag{2}$$

This defines equilibrium wage w^*. For a stationary equilibrium, a measure of firms should be invariant. Let μ^* be such a measure, then it satisfies

$$\mu^*([z^*,s]) = \lambda\Big[G(s) - G(z^*)\Big] + \int \Big[F(s|z) - F(z^*|z)\Big]\mu^*(dz)$$

λ characterizes μ^* according to the above equation. λ is defined by the following market clearing condition.

$$\int_{z^*} zg(n)\mu^*(dz) = w^* \tag{3}$$

LHS is the total output produced by all the existing firms, and RHS is the wage the representative agent gets. It is not difficult to modify this condition by defining the number of consumers. Equations (1), (2), and (3) define a stationary equilibrium.

Now consider θ economy where productivity is θz for all z. Firm's profit function is defined by

$$\pi_\theta(z,w) = \max_n \theta z g(n) - w(n+f)$$

Note that

$$\pi_\theta(z,w) = \pi(\theta z, w)$$

Also a value function is defined by

$$V_\theta(z,w) = \max\left\{0\ ,\ \pi_\theta(z,w) + \beta\int V_\theta(z',w)F(dz'|z)\right\}$$

A firm exits if it gets $z < z^*_\theta$ where z^*_θ is a threshold satisfying

$$\pi_\theta(z^*_\theta,w) + \beta\int V_\theta(z',w)F(dz'|z^*_\theta) = 0 \tag{4}$$

An entrant has a value

$$V^e_\theta(w) = \int V_\theta(z,w)G(dz) - c_e w$$

Therefore, the following equation holds

$$\int V_\theta(z,w)G(dz) = c_e w \tag{5}$$

Let μ^*_θ be an invariant measure, then

$$\mu^*_\theta([z^*_\theta, s]) = \lambda_\theta\Big[G(s) - G(z^*_\theta)\Big] + \int\Big[F(s|z) - F(z^*_\theta|z)\Big]\mu^*_\theta(dz)$$

λ_θ is defined by the following market clearing condition.

$$\int_{z^*_\theta} \theta z g(n)\mu^*_\theta(dz) = w^*_\theta \tag{6}$$

Equations (4), (5), and (6) define a stationary equilibrium in θ economy.

CLAIM *Between the stationary equilibria in the two economies, the following condition holds.*

$$w^*_\theta = \theta w^*\ ,\ z^*_\theta = z^*\ ,\ \lambda_\theta = \lambda$$

To prove the claim, let w^*, z^* and λ satisfy equations (1), (2), and (3). Check if $w^*_\theta = \theta w^*, z^*_\theta = z^*$, and $\lambda_\theta = \lambda$ satisfy equations (4), (5), and (6). Note first that the following holds.

$$\begin{aligned}
\pi_\theta(z,w^*_\theta) &= \max_n \theta z g(n) - w^*_\theta(n+f)\\
&= \max_n \theta z g(n) - \theta w^*(n+f)\\
&= \theta\Big[\max_n z g(n) - w^*(n+f)\Big]\\
&= \theta\pi(z,w^*)
\end{aligned}$$

Given this, guess a value function as

$$V_\theta(z, w_\theta^*) = \theta V(z, w^*)$$

then, it satisfies the defintion of V_θ. To show that w_θ^* and z_θ^* satisfy (4), evaluate (4) at w_θ^* and rewrite it using the above two equations as

$$\theta\pi(z_\theta^*, w^*) + \int \theta V(z', w^*)F(dz'|z_\theta^*) = 0$$

This holds for $z_\theta^* = z^*$ by (1). In the same way, (5) is equivalent to

$$\int \theta V(z, w^*)G(dz) = c_e w_\theta^*$$

which holds for $w_\theta^* = \theta w^*$ by (2). Now from invariant condition, $\lambda_\theta = \lambda$ guarantees that $\mu_\theta^* = \mu^*$. Therefore (6) holds for $z_\theta^* = z^*$ and $w_\theta^* = \theta w^*$ by (3). This completes the proof that $w_\theta^* = \theta w^*, z_\theta^* = z^*$, and $\lambda_\theta = \lambda$ consistitute a stationary equilibrium.[1]

[1] To prove that those are a unique stationary equilibrium, note first that $w_\theta^* = \theta w^*$ leads to $z_\theta^* = z^*$ and $\lambda_\theta = \lambda$ by the same argument. Consider $w_\theta^* < \theta w^*$. Denote n^* and n_θ^* be optimal labor input in the original economy and in θ economy, respectively. Since

$$\pi_\theta(z, w_\theta^*) = \theta z g(n_\theta^*) - w_\theta^*(n_\theta^* + f) \geq \theta z g(n^*) - w_\theta^*(n^* + f) > \theta z g(n^*) - \theta w^*(n^* + f) = \theta\pi(z, w^*)$$

we have $V_\theta(z, w_\theta^*) > \theta V(z, w^*)$, which leads to $z_\theta^* < z^*$ from (4). But this contradicts to (5) since (2) implies

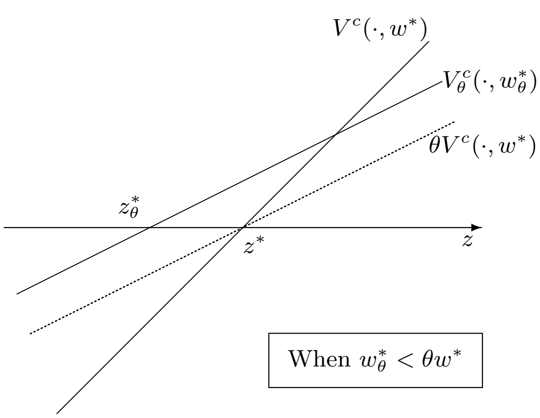
$$\int_{z_\theta^*} V_\theta(z, w_\theta^*)G(dz) > \int_{z^*} \theta V(z, w^*)G(dz) = c_e\theta w^* > c_e w_\theta^*$$

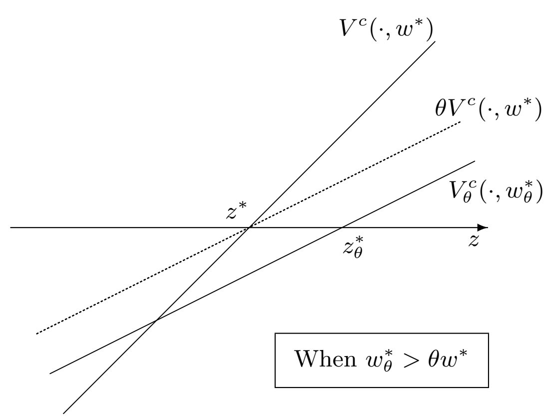
Next consider $w_\theta^* > \theta w^*$. Since

$$\pi_\theta(z, w_\theta^*) = \theta z g(n_\theta^*) - w_\theta^*(n_\theta^* + f) < \theta z g(n_\theta^*) - \theta w^*(n_\theta^* + f) \leq \theta[z g(n^*) - w^*(n^* + f)] = \theta\pi(z, w^*)$$

we have $V_\theta(z, w_\theta^*) < \theta V(z, w^*)$, which leads to $z_\theta^* > z^*$ from (4). But this contradicts to (5) since (2) implies

$$\int_{z_\theta^*} V_\theta(z, w_\theta^*)G(dz) < \int_{z^*} \theta V(z, w^*)G(dz) = c_e\theta w^* < c_e w_\theta^*$$





Remark: V^c is a continuing value such that $V(z, w) = \max\{0\ ,\ V^c(z, w)\}$. Note that V_θ^c and θV^c may not be parellel to each other.

(b)

A dynamic programming problem a firm faces is defined as follows.

$$V(z,w) = \max\left\{0\ ,\ \max_x \left[\pi(z,w) - xw + \beta \int V(z',w)F(dz'|z,x)\right]\right\}$$

At the stationary equilibrium, FOC reads as

$$w^* = \beta \int V(z',w^*)F_3(dz'|z,x^*)$$

This equation defines optimal investment $x^*(z)$ for all z. The following conditions as well as the above one define the stationary equilibrium.

$$\pi(z^*,w^*) - x^*(z^*)w^* + \beta \int V(z',w^*)F(dz'|z^*,x^*(z^*)) = 0 \qquad \text{(threshold)}$$

$$\int V(z,w^*)G(dz) = c_e w^* \qquad \text{(entry)}$$

$$\int_{z^*} zg(n)\mu^*(dz) = w^* \qquad \text{(invariance)}$$

Consider θ economy. A value function is defined by

$$V_\theta(z,w) = \max\left\{0\ ,\ \max_x \left[\pi_\theta(z,w) - xw + \beta \int V_\theta(z',w)F(dz'|z,x)\right]\right\}$$

At the stationary equilibrium, FOC reads as

$$w_\theta^* = \beta \int V_\theta(z',w_\theta^*)F_3(dz'|z,x_\theta^*)$$

The following conditions as well as the above one define the stationary equilibrium.

$$\pi_\theta(z_\theta^*,w_\theta^*) - x_\theta^*(z_\theta^*)w_\theta^* + \beta \int V_\theta(z',w_\theta^*)F(dz'|z_\theta^*,x_\theta^*(z_\theta^*)) = 0 \qquad \text{(threshold)}$$

$$\int V_\theta(z,w_\theta^*)G(dz) = c_e w_\theta^* \qquad \text{(entry)}$$

$$\int_{z_\theta^*} \theta zg(n)\mu_\theta^*(dz) = w_\theta^* \qquad \text{(invariance)}$$

Now comparing the stationary equilibria (z^*,w^*,λ,x^*) to $(z_\theta^*,w_\theta^*,\lambda_\theta,x_\theta^*)$, there is the following relationship.

$$w_\theta^* = \theta w^*\ ,\ \ z_\theta^* = z^*\ ,\ \ \lambda_\theta = \lambda\ ,\ \text{ and } x_\theta^*(z) = x^*(z)\ \forall z$$

The proof uses the same method used in (a). Let (z^*,w^*,λ,x^*) be the stationary equilibrium in the original economy. In other words, they satisfy the four conditions above in the original economy. Note that when $\lambda_\theta = \lambda$, we have $\mu_\theta^* = \mu^*$. Also from $w_\theta^* = \theta w^*$, we have $\pi_\theta(z,w_\theta^*) = \theta\pi(z,w^*)$, and $V_\theta(z,w_\theta^*) = \theta V(z,w^*)$. Then, $(z_\theta^*,w_\theta^*,\lambda_\theta,x_\theta^*)$ defined above satisfy the four conditions above in θ economy. Therefore, x_θ^* does not depend on θ. Regardless of θ, x_θ^* is defined by $x_\theta^*(z) = x^*(z)$ for all z.

Industry equilibrium: Monopolistic competition

16. (a)

Let $Q(\mu)$ be the production function of the final good that has the following form

$$Q(\mu) = \left(\int q(z)^{\frac{\sigma-1}{\sigma}} \mu(dz) \right)^{\frac{\sigma}{\sigma-1}}$$

Also, its price is defined as[2]

$$P(\mu) = \left(\int p(z)^{1-\sigma} \mu(dz) \right)^{\frac{1}{1-\sigma}}$$

The final good firm wants to maximize its profit.

$$\max_{q(z)} PQ - \int p(z)q(z)\mu(dz)$$

FOC writes as

$$PQ^{\frac{1}{\sigma}} q(z)^{-\frac{1}{\sigma}} - p(z) = 0$$

or, equivalently

$$q(z) = p(z)^{-\sigma} P^{\sigma} Q$$

This is the demand for the commodity which firm z produces. So firm z solves

$$\max_{q(z)} p(z)q(z) - cq(z)$$

where c is the marginal cost of firm z which is an idiosyncratic shock. Optimal markup pricing rule yields

$$p(z) = \left(1 - \frac{1}{\epsilon_d}\right)^{-1} c = \frac{\sigma}{\sigma - 1} c$$
$$q(z) = p(z)^{-\sigma} P^{\sigma} Q = \left(\frac{\sigma - 1}{\sigma c} \right)^{\sigma} P^{\sigma} Q$$

[2] This form can be derived by minimizing cost to produce Q units of output.

$$\min_{q(z)} \int p(z)q(z)\mu(dz) \quad \text{s.t.} \quad \left(\int q(z)^{\frac{\sigma-1}{\sigma}} \mu(dz) \right)^{\frac{\sigma}{\sigma-1}} = Q$$

With Largangian multiplier λ, FOC writes as

$$-p(z) + \lambda \left(\int q(z)^{\frac{\sigma-1}{\sigma}} \mu(dz) \right)^{\frac{1}{\sigma-1}} q(z)^{-\frac{1}{\sigma}} = 0$$

for all z. Simplify FOC using the constraint as $p(z) = \lambda Q^{\frac{1}{\sigma}} q(z)^{-\frac{1}{\sigma}}$, and substitute back into the constraint, then

$$\left(\int \left[\lambda^{\sigma} p(z)^{-\sigma} Q\right]^{\frac{\sigma-1}{\sigma}} \mu(dz) \right)^{\frac{\sigma}{\sigma-1}} = Q$$

Arranging terms,

$$\lambda = \left(\int p(z)^{1-\sigma} \mu(dz) \right)^{\frac{1}{1-\sigma}}$$

This is the shadow price of Q.

So its profit is

$$\text{profit} = [p(z) - c]q(z) = \frac{c}{\sigma - 1}\left(\frac{\sigma - 1}{\sigma c}\right)^{\sigma} P^{\sigma} Q = c^{1-\sigma}\left[\frac{(\sigma - 1)^{\sigma - 1}}{\sigma^{\sigma}}\right] P^{\sigma} Q$$

Define

$$z \equiv c^{1-\sigma}$$
$$\pi(\mu) \equiv \frac{(\sigma - 1)^{\sigma - 1}}{\sigma^{\sigma}} P(\mu)^{\sigma} Q(\mu)$$

then we can write firm z's profit as $z\pi(\mu)$.

Also in the **perfectly competitive market**, a firm's profit can be written in this form. Note that $Q(\mu) = \int q(z)\mu(dz)$ and $P(\mu) = D(Q)$ for some inverse demand function D. Let firm z has the cost function $c(q) = cq^2$. Firm z solves

$$\max Pq - cq^2$$

FOC yields $q = \frac{P}{2c}$, and thus its profit is $\frac{P^2}{4c}$. Define $z \equiv \frac{1}{4c}$ and $\pi(z) \equiv P(\mu)^2$, then we can write firm z's profit as $z\pi(\mu)$.

(b)

A firm's value can be written as

$$V(z, \mu) = \max_x \left\{ z\pi(\mu) - zc(x, \mu) + \beta \int V(z\varepsilon, \mu')G(d\varepsilon, x) \right\}$$

since there is no exit. μ' is defined so that it is a measure of $z\varepsilon$, where z is distributed with measure μ and ε has a cdf $G(\varepsilon, x)$. Consider the stationary equilibrium where $\mu' = \mu$. Guess a value function as $V(z, \mu) = zv(\mu)$, where $v(\mu)$ depends only on μ. Then, the above equation can be written as

$$zv(\mu) = \max_x \left\{ z\pi(\mu) - zc(x, \mu) + \beta \int z\varepsilon v(\mu)G(d\varepsilon, x) \right\}$$

By defining $v(\mu)$ as a value function satisfying

$$v(\mu) = \max_x \left\{ \pi(\mu) - c(x, \mu) + \beta v(\mu) \int \varepsilon G(d\varepsilon, x) \right\}$$

we can write $V(z, \mu) = zv(\mu)$. Also optimal choice of x depends only on μ, the measure of shocks, but not on z, the idiosyncratic shock each firm gets. So all firms choose the same level of investment.

Atkeson and Burstein analyze the impact of decline in the marginal cost on innovation. They insist that under some conditions, decline in the marginal cost leads to an increase in the profit of all firms by the same proportion, and thus that the level of innovation does not change. The model in this question supports their claim in the sense that a different cost shock does not lead to different level of investment. Also, Atkeson and Burstein argue that under other conditions, changes in marginal cost increase some firms' profit by more proportion than others' profit, and thus that the former firms invest more. It is also consistent with this model. However, Atkeson and Burstein consider the case where the marginal cost of all firms changes by the same proportion. In their paper, it could be that different firms choose different level of innovation. In this question, all firms choose the same level of investment. On the other hand, if the marginal cost of all firms changes by the same proportion, the distribution of shocks changes, and thus affects the level of investment.

(c)

If firm z's profit is $z\pi(\mu) - f$ instead of $z\pi(\mu)$, the argument in (b) does not hold. The value function is not multiplicative in z. So the level of investment depends on z as well as on μ. Because of fixed cost, firm's profit increases faster than a shock it gets. Therefore, a firm whose shock is large invests more.

Industry equilibrium: Dynamics

18. (a)

Define n_t^1 as the number of firms which have $\theta = 1$ at period t. Also n_t^0 is the number of firms which have $\theta = 0$ at period t. Let $\alpha \in (0, 1)$ be the probability that firms produce θ. Since α portion of n_t^1 firms would produce 1 at t, it should be satisfied that

$$p_t = D(\alpha n_t^1) \tag{7}$$

Let $\overline{v}$ be the value of a firm that has learned to have $\theta = 1$, and v_s^c be the continuing value of a firm that entered s periods ago and had all zero output. A firm would exit if $v_s^c < 0$. There would be some s^* such that

$$v_{s^*-1}^c \geq 0 \ , \ v_{s^*}^c < 0 \tag{8}$$

Also, an entrant should be indifferent between entering and staying out.

$$v^e = \lambda\alpha(p - f + \beta\overline{v}) + (1 - \lambda\alpha)(-f + \beta v_1) - c_e = 0 \tag{9}$$

(8) defines s^*, and thus which firm exits. (9) defines p_t. (7) defines n_t^1. Finally, n_t^0 is defined by the following equations.

$$n_{t+1}^1 = (1 - \delta_t^1)n_t^1 + \gamma_{t+1}\lambda \tag{10}$$
$$n_{t+1}^0 = (1 - \delta_t^0)n_t^0 + \gamma_{t+1}(1 - \lambda) \tag{11}$$

where γ_t is the number of firms entering at period t, and δ_t^1 and δ_t^0 are the portion of firms exiting after period t ends, which are functions of s^* and history of $(n_0^1, \cdots, n_t^1)$ and $(n_0^0, \cdots, n_t^0)$. γ_t is defined by (10), and n_t^0 is defined by (11). So equations (7) to (11) define an industry equilibrium.

(b)

Note that there is no t in (9). Since p_t should be set so that an entrant is indifferent between entering and staying out at any period, p_t should be constant over t. Let $p_t = p$. Since p_t is constant, the continuing firm's value v^c_s depends only on s. Therefore s^* does not depend on t. Suppose that there exists no firm at period 0. So $n^1_0 = n^0_0 = 0$. The measure of firms are uniquely determined from (7), (10), and (11).

Consider period 1. From (7), n^1_1 is set so that $p = D(\alpha n^1_1)$. From (10), γ_1 is set so that $n^1_1 = \gamma_1\lambda$. From (11), n^0_1 is set so that $n^0_1 = \gamma_1(1-\lambda)$. In other words, at period 1, γ_1 firms enter. λ portion of them have $\theta = 1$, and α portion of those produce 1 unit of output. $1-\alpha$ portion do not produce. $1-\lambda$ portion of entering firms have $\theta = 0$ and do not produce. So the total supply is $\lambda\alpha\gamma_1$, and it should be that $p = D(\lambda\alpha\gamma_1)$. These are uniquely determined from equations in (a).

Consider period 2. If $s^* > 1$, all firms stay in the market. In this case, $\delta^1_1 = \delta^0_1 = 0$. From $p = D(\alpha n^1_2)$, we have $n^1_2 = n^1_1$. Since no firm exits, $\gamma_2 = 0$ from (10) and thus $n^0_2 = n^0_1$ from (11). These are uniquely determined. If $s^* = 1$, all firms that do not produce at period 1 exit before period 2 begins. Again, $n^1_2 = n^1_1$, so the same number of type 1 firms should enter as the number of firms that exit. γ_2 is defined to satisfy this. Formally,

$$n^1_2 = (1-\alpha)n^1_1 + \gamma_2\lambda$$

To satisfy $n^1_2 = n^1_1$,

$$\gamma_2 = \frac{\alpha n^1_1}{\lambda} = \alpha\gamma_1$$

From this, $n^0_2 = \gamma_2(1-\lambda)$. We can see that all of these are uniquely determined. The same logic applies to the periods thereafter.

(c)

The value of a firm that has learned that $\theta = 1$ is

$$\overline{v} = \alpha p - f + \beta\overline{v}$$

or equivalently,

$$\overline{v} = \frac{\alpha p - f}{1-\beta}$$

The value of a firm that entered t periods ago and had all zero output consists of two parts. If the firm produces 0, its value would be $-f + \beta v_{t+1}$. If the firm produces 1, its value would be $p - f + \beta\overline{v}$. The probability that the firm is type 1 given that it produces nothing for t periods is, by Bayes rule,

$$\eta_t \equiv \frac{\Pr(\text{type 1})\Pr(\text{no output for } t \mid \text{type 1})}{\Pr(\text{no output for } t)} = \frac{\lambda(1-\alpha)^t}{\lambda(1-\alpha)^t + 1 - \lambda}$$

So v_t is

$$v_t = \max\Big\{0\ ,\ \alpha\eta_t(p - f + \beta\overline{v}) + (1-\alpha\eta_t)(-f + \beta v_{t+1})\Big\}$$

(d)

At the stationary equilibrium, n_t^1 and n_t^0 should be invariant. As we saw in (b), n_t^1 is always invariant with $p = D(\alpha n_t^1)$. But for n_t^0 to be invariant over time, it should be that all firms of type 1 in the market have learned $\theta = 1$. Therefore there is no entry and exit and $n_t^0 = 0$.

To see this, suppose there are firms that did not learn $\theta = 1$ until t. Some periods later, there would still exist firms that did not learn $\theta = 1$, since $\alpha < 1$. So those firms eventually exit at some period. It makes new firms enter so that n_t^1 should be invariant. This process will be repeated infinitely, but the number of exiting firms and that of entering firms will decrease as time goes by, because at each period, some fraction of type 1 firms would learn that they are type 1. This is not a stationary equilibrium.

A stationary equilibrium is (p^*, n_t^1, n_t^0) that satisfy

$$\begin{aligned} p &= D(\alpha n_t^1) \\ n_t^0 &= 0 \\ c_e &= \lambda\alpha(p - f + \beta\overline{v}) + (1-\lambda\alpha)(-f + \beta v_1) \end{aligned}$$

s^* does not affect the equilibrium, since there is no entry and exit. Accordingly $\delta_t^1 = \delta_t^0 = \gamma_t = 0$.

(e)

From (c), we have $\overline{v}$ and v_t, which are functions of p. From (9), we have

$$\lambda\alpha(p - f + \beta\overline{v}) + (1-\lambda\alpha)(-f + \beta v_1) = 0$$

with which we can solve for p. At period 0, there is no firm. At period 1, γ_1 firms enter where

$$\begin{aligned} \text{the number of type 1 firms: } & n_1^1 = \lambda\gamma_1 \\ \text{the number of type 0 firms: } & n_1^0 = (1-\lambda)\gamma_1 \\ \text{EQ condition: } & \quad p = D(\alpha n_1^1) \end{aligned}$$

All firms stay and no firm enters until period s^*. After period s^* ends, $(1-\alpha)^{s^*}$ portion of type 1 firms do not learn $\theta = 1$. Also all type 0 firms do not learn $\theta = 1$. All of those firms exit before period $s^* + 1$ begins. The number of type 1 firms should be constant, so γ_{s^*+1} firms enter where

$$\begin{aligned} \text{the number of type 1 firms: } & n_{s^*+1}^1 = [1 - (1-\alpha)^{s^*}]n_1^1 + \lambda\gamma_{s^*+1} \\ \text{the number of type 0 firms: } & n_{s^*+1}^0 = (1-\lambda)\gamma_{s^*+1} \\ \text{EQ condition: } & n_{s^*+1}^1 = n_1^1 \end{aligned}$$

As we have seen, n_t^1 is constant over t. n_t^0 is constant for $t = 1, \cdots, s^*$, but it becomes less at s^*+1. It is because $\gamma_{s^*+1} < \gamma_1$ from the above equations. But n_t^0 is constant again for $t = s^*+1, \cdots, 2s^*$. So the total number of firms is constant for $t = 1, \cdots, s^*$, but becomes less and stays the same for $t = s^* + 1, \cdots, 2s^*$. This process repeats infinitely. The total number of firms would weakly decrease over time and converge to n_t^1 as $t \to \infty$, but it never achieves n_t^1.

19. (a)

An existing firm has the following value

$$V_t(\gamma_t, Q_t) = \max\left\{0\ ,\ \max_{\{q_s\}_{s=t}^{\infty}} \sum_{s=t}^{\infty} \beta^{s-t} E\left[D\left(\frac{Q_s}{\gamma_s}\right) q_s - c(q_s) \middle|\, \gamma_t, Q_t \right]\right\}$$

Write this as a dynamic programming problem.

$$V_t(\gamma_t, Q_t) = \max\left\{0\ ,\ \max_{q_t}\left\{ D\left(\frac{Q_t}{\gamma_t}\right) q_t - c(q_t) + \beta E[V_{t+1}(\gamma_{t+1}, Q_{t+1})|\gamma_t, Q_t]\right\}\right\}$$

Define its continuing value as $V_t^c(\gamma_t, Q_t)$. If $V_t^c(\gamma_t, Q_{t-1}) < 0$, then some firms would exit and there will be no entry. In this case, the following equation defines Q_t and thus $n_t = \frac{Q_t}{q_t}$, the number of firms at period t.

$$V_t^c(\gamma_t, Q_t) = 0 \tag{12}$$

If $V_t^c(\gamma_t, Q_{t-1}) > 0$, then no firm exits. There are two subcases by an entrant. An entrant has the following value

$$V_t^e(\gamma_t, Q_t) = V_t(\gamma_t, Q_t) - c_e$$

Since an entrant should weakly prefer staying out to entering,

$$V_t(\gamma_t, Q_t) \leq c_e \tag{13}$$

with equality when there is entry. If $V_t^c(\gamma_t, Q_{t-1}) < c_e$, there is no entry, either. Otherwise, $V_t(\gamma_t, Q_t) = c_e$ defines Q_t, and thus n_t. A competitive equilibrium is a sequence of variables (Q_t, q_t, p_t, n_t) that satisfy the above condition and

$$\begin{aligned}
q_t &= \arg\max p_t q_t - c(q_t) + \beta E[V_{t+1}(\gamma_{t+1}, Q_{t+1})|\gamma_t, Q_t] \\
p_t &= D\left(\frac{Q_t}{\gamma_t}\right) \\
n_t &= \frac{Q_t}{q_t}
\end{aligned}$$

Therefore q_t, Q_t, and p_t would be defined uniquely from the above process, and so would the number of firms.

(b)

Let $\gamma_t = (1+g)\gamma_{t-1}$. In other words, the industry is in the growing stage. With probability ρ, $\gamma_{t+1} = (1-\delta)\gamma_t$, and with probability $1-\rho$, $\gamma_{t+1} = (1+g)\gamma_t$. Therefore

$$V_t(\gamma_t, Q_t) = \max_{q_t}\left\{ D\left(\frac{Q_t}{\gamma_t}\right) q_t - c(q_t) + \beta\Big[(1-\rho)V_{t+1}([1+g]\gamma_t, Q^*) + \rho V_{t+1}([1-\delta]\gamma_t, Q^*)\Big]\right\}$$

where Q^* is an expected future total output that satisfies either (12) or (13) under each situation. If γ_t grows at the rate g, the current value is at least as large as the previous value. So, according to (13), $V_{t+1}([1+g]\gamma_t, Q^*) = V_t(\gamma_t, Q_t) = c_e$. If γ_t falls at the rate δ, the current value is less than the previous value, and $0 \leq V_{t+1}([1-\delta]\gamma_t, Q^*) < c_e$ according to (12) and (13). Call the value v_d. Using these conditions, we can write the above equation as follows.

$$c_e = \max_{q_t} \left\{ D\left(\frac{Q_t}{\gamma_t}\right) q_t - c(q_t) + \beta[(1-\rho)c_e + \rho v_d] \right\} \tag{14}$$

Since the same form of equation would hold at every period in the growing stage, we would have

$$D\left(\frac{Q_t}{\gamma_t}\right) = D\left(\frac{Q_{t-1}}{\gamma_{t-1}}\right)$$
$$q_t = q_{t-1}$$

So in the growing stage, each firm's output is constant over time, and the equilibrium price is also constant over time. Since $\gamma_t = (1+g)\gamma_{t-1}$, it follows that $Q_t = (1+g)Q_{t-1}$, and thus that $n_t = (1+g)n_{t-1}$. In other words, the number of firms grows at the same rate with demand.

Now consider the decaying stage. Suppose t^* is the first period when γ_t decreases. Since the switch is permanent,

$$v_d = V_{t^*}(\gamma_{t^*}, Q_{t^*}) = \max_{q_{t^*}} \left\{ D\left(\frac{Q_{t^*}}{\gamma_{t^*}}\right) q_{t^*} - c(q_{t^*}) + \beta V_{t^*+1}([1-\delta]\gamma_{t^*}, Q^*) \right\}$$

There are two subcases.

(I) If $v_d = 0$, $0 \leq V_{t^*+1}([1-\delta]\gamma_{t^*}, Q^*) \leq V_{t^*}(\gamma_{t^*}, Q_{t^*}) = 0$, so we can write the above equation as

$$0 = \max_{q_{t^*}} D\left(\frac{Q_{t^*}}{\gamma_{t^*}}\right) q_{t^*} - c(q_{t^*}) \tag{15}$$

Clearly, it cannot be $D\left(\frac{Q_{t^*-1}}{\gamma_{t^*-1}}\right) = D\left(\frac{Q_{t^*}}{\gamma_{t^*}}\right)$, because if the prices are the same under both stages, the maximum profit would be 0 at period t^*-1, which is a contradiction to (14). So it must be that $D\left(\frac{Q_{t^*-1}}{\gamma_{t^*-1}}\right) > D\left(\frac{Q_{t^*}}{\gamma_{t^*}}\right)$. Q_{t^*} would fall at a smaller rate than δ. q_{t^*} decreases as well because of decline in price. So the number of firms decreases at a smaller rate than Q_{t^*} decreases.

(II) If $v_d > 0$, no existing firm exits. Suppose $D\left(\frac{Q_{t^*-1}}{\gamma_{t^*-1}}\right) = D\left(\frac{Q_{t^*}}{\gamma_{t^*}}\right)$. Then, profit maximizing q_{t^*} should be the same with q_{t^*-1}. But since no firm exits, this implies that $Q_{t^*} = Q_{t^*-1}$, and thus that $\frac{Q_{t^*-1}}{\gamma_{t^*-1}} < \frac{Q_{t^*}}{\gamma_{t^*}}$ which is a contradiction. So it must be that $D\left(\frac{Q_{t^*-1}}{\gamma_{t^*-1}}\right) > D\left(\frac{Q_{t^*}}{\gamma_{t^*}}\right)$. Again Q_{t^*} would fall at a smaller rate than δ. q_{t^*} decreases as well because of decline in price. So the number of firms decreases at a smaller rate than Q_{t^*} decreases. This repeats until firm's value becomes 0.

But in both cases, at some periods later, firm's value may become 0, so solve the same problem with (15). Once the industry enters in the stage where γ_t falls and firm's value is 0,

$$D\left(\frac{Q_t}{\gamma_t}\right) = D\left(\frac{Q_{t+1}}{\gamma_{t+1}}\right) \quad \text{and} \quad q_t = q_{t+1}$$

In this stage, therefore, each firm's output and the equilibrium price are constant over time, and Q_t will decrease at the rate δ. So the number of firms decreases at the rate δ as well.

(c)

According to the result of (b), in the growing stage, the growth rate of n_t is the same with that of Q_t, while in the decaying stage, the depreciation rate of n_t is less than that of Q_t for some periods after switch. This supports the empirical evidence given in the question, if we assume that the size of employment is proportional to the total output. We can test if price and each firm's output decrease in the decaying stage. It is also testable if price stops falling as soon as some firms exit.

(d)

If outside option is $\phi > 0$, all existing firms should have a value at least as big as ϕ. (15) changes to

$$(1-\beta)\phi = \max_{q_{t^*}} D\left(\frac{Q_{t^*}}{\gamma_{t^*}}\right) q_{t^*} - c(q_{t^*})$$

but the essence of dynamics would be similar. In the decaying stage, firms starts exiting earlier as ϕ increases. If $\phi = c_e$, then firm's value is always c_e, so firms would solve the same form of the problem at any period. Therefore, as soon as the industry enters in the decaying stage, firms would exit at the same rate at which demand shrinks and total output decreases.

(e)

If firms have different cost, firms with low cost will grow fast in the growing stage. So those firms will have bigger size than others. Firms with high cost will exit earlier than those with low cost in the decaying stage. Since the firms with high cost would have produced less than others, the number of firms would decrease faster than in the previous case. So now we cannot tell which would decrease faster between the number of firms and the total output. But firms with bigger size would last longer and reduce its output at a smaller rate.

20. (a)

Let α be the probability that firms with capacity 1 successfully increase their capacity to 2. Productive firms have the following value.

$$V_t^2(p_t) = \max\Big\{0\ ,\ 2p_t - c + \beta E V_{t+1}^2(p_{t+1})\Big\}$$

Also less productive firms have the following value.

$$V_t^1(p_t) = \max\Big\{0\ ,\ p_t - c + \beta E V_{t+1}^1(p_{t+1})\ ,\ \ -x + E\Big[\alpha V_{t+1}^2(p_{t+1}) + (1-\alpha)[p_t - c + \beta V_{t+1}^1(p_{t+1})]\Big]\Big\}$$

There would be no exit of firms with capacity 2. If there is exit, it should be from firms with capacity 1. They would exit if $p_t < p^*$ where

$$p^* = \sup\{p_t : V_t^1(p_t) = 0\}$$

They would invest in R&D if $p > p^*$ and $p > p^{**}$ where

$$p^{**} - c + \beta E V_{t+1}^1(p_{t+1}) = -x + E\Big[\alpha V_{t+1}^2(p_{t+1}) + (1-\alpha)[p^{**} - c + \beta V_{t+1}^1(p_{t+1})]\Big]$$

An entrant would consider the following value.

$$V_t^e(p_t) = V_t^1(p_t) - c_e$$

Suppose there is no firm in period 0. In the first period, there would be entry if we assume that demand is high enough. An entrant should be indifferent between entering and staying out. So p_1 would be determined so that

$$V_1^1(p_1) = c_e \tag{16}$$

The number of firms entering would be determined by the following equation.

$$D(p_1) = n_1 \tag{17}$$

There is no more entry from period 2. There are two possible stories. One is that no firm invests, and that all firms would permanently have capacity 1. Then, in the following periods, the existing firms produce n_1 units of output in total, which supports $p_t = p_1$. So the price does not change and no firm enters. The other is that all firms invest, and that some of them would have capacity 2. Then, in the following periods, the existing firms produce more than n_1 units of output in total, which makes $p_t < p_1$ for $t \geq 2$. So for $t \geq 2$,

$$V_t^e(p_t) = V_t^1(p_t) - c_e < V_1^1(p_1) - c_e = 0$$

and thus no firm enters.

(b)

When no firm exits and all firms have capacity 2 in the limit, the supporting price $\overline{p}$ is

$$D(\overline{p}) = 2n_1$$

where n_1 is defined by (16) and (17). So if a firm with capacity 1 wants to invest even under $\overline{p}$, then it is an equilibrium that no firm exits and all firms have capacity 2 in the limit. Consider a hypothetical firm with capacity 1 when all the firms have capacity 2. The price in the next period will be $\overline{p}$, too. So the condition that a firm with capacity 1 still wants to invest under $\overline{p}$ is

$$\overline{p} - c + \beta V^1(\overline{p}) \leq -x + \alpha V^2(\overline{p}) + (1-\alpha)[\overline{p} - c + \beta V^1(\overline{p})] = V^1(\overline{p})$$

Note that under $\overline{p}$,

$$V^2(\overline{p}) = 2\overline{p} - c + \beta V^2(\overline{p})$$

or equivalently,

$$V^2(\overline{p}) = \frac{2\overline{p} - c}{1-\beta}$$

Plug this back into the second equality of the above condition, then we can get $V^1(\overline{p})$ as

$$V^1(\overline{p}) = \frac{-x + \alpha V^2(\overline{p}) + (1-\alpha)(\overline{p} - c)}{1 - \beta(1-\alpha)} = \frac{\overline{p} - c}{1-\beta} + \frac{\alpha\overline{p} - x(1-\beta)}{[1-\beta(1-\alpha)](1-\beta)}$$

The inequality of the above condition is equivalent to

$$\frac{\overline{p}-c}{1-\beta} \leq V^1(\overline{p})$$

Combine these two to get

$$0 \leq \alpha\overline{p} - x(1-\beta)$$

This can be written as

$$x \leq \frac{\alpha\overline{p}}{1-\beta}$$

Note that x is an investment cost, and RHS is the expected increase of the profit by investment. In addition, we need a trivial condition that $V^1(\overline{p}) \geq 0$, which is equivalent to

$$(\overline{p}-c)[1-\beta(1-\alpha)] + \alpha\overline{p} - x(1-\beta) \geq 0$$

Therefore no firm exits and all firms have capacity 2 in the limit if

$$x \leq \min\left\{\frac{\alpha\overline{p}}{1-\beta}\ ,\ \frac{\alpha\overline{p}+(\overline{p}-c)[1-\beta(1-\alpha)]}{1-\beta}\right\}$$

(c)

If $V^1(\overline{p}) < 0$, then some firms would exit at some point. There exists $\widehat{n} \in (n_1, 2n_1)$ such that $D(\widehat{p}) = \widehat{n}$, and when $p_t > \widehat{p}$, firms want to invest, but when $p_t < \widehat{p}$, firms want to exit. So under $\widehat{p}$, firms with capacity 1 should be indifferent between investing and exiting and the value should be greater than what they can get by staying and not investing.

$$V^1(\widehat{p}) = -x + \alpha V^2(\widehat{p}) + (1-\alpha)[\widehat{p} - c + \beta V^1(\widehat{p})] = 0 \geq \widehat{p} - c + \beta V^1(\widehat{p})$$

Again, $V^2(\widehat{p}) = \frac{2\widehat{p}-c}{1-\beta}$, so from the first and the second equalities,

$$x = \alpha V^2(\widehat{p}) + (1-\alpha)[\widehat{p} - c + \beta V^1(\widehat{p})] = \frac{\alpha\widehat{p}+(\widehat{p}-c)[1-\beta(1-\alpha)]}{1-\beta}$$

Also from the last inequality, $\widehat{p} \leq c$, so we obtain

$$x \leq \frac{\alpha c}{1-\beta}$$

This condition together with $V^1(\overline{p}) < 0$ defines the following condition.

$$\frac{\alpha\overline{p}+(\overline{p}-c)[1-\beta(1-\alpha)]}{1-\beta} < x \leq \frac{\alpha c}{1-\beta}$$

If this condition holds, the following occurs. Starting from p_1, firms invest and some of them increase their capacity. At some period, too many firms have capacity 2, so firms with capacity 1 would have negative value if no firm exits. Therefore some firms exit so that the total output becomes $\widehat{n}$, and the price becomes $\widehat{p}$. Thereafter firms with capacity 1 invest, and as some of them successfully increase their capacity, those who fail may exit so that price stays at $\widehat{p}$. This repeats either infinitely or until all unproductive firms exit.