2000 Spring 1. Wealth Accumulation

(a) By definition,

$$\begin{aligned} W_{t+1} &= K_{t+1} + y_{t+1} + \frac{y_{t+2}}{1+r} + \frac{y_{t+2}}{(1+r)^2} + \cdots \\ &= (1+r)(K_t + y_t - C_t) + y_{t+1} + \frac{y_{t+2}}{1+r} + \frac{y_{t+2}}{(1+r)^2} + \cdots \\ &= (1+r)\left[K_t + y_t + \frac{y_{t+1}}{1+r} + \frac{y_{t+2}}{(1+r)^2} + \frac{y_{t+2}}{(1+r)^3} + \cdots\right] - (1+r)C_t \\ &= (1+r)(W_t - C_t) \end{aligned}$$

(b) Form Lagrangian

$$\mathcal{L} = U(C) + \sum_{t=1}^{T} \lambda_t \left[(1+r)(W_t - C_t) - W_{t+1}\right]$$

Thus we have FOC's

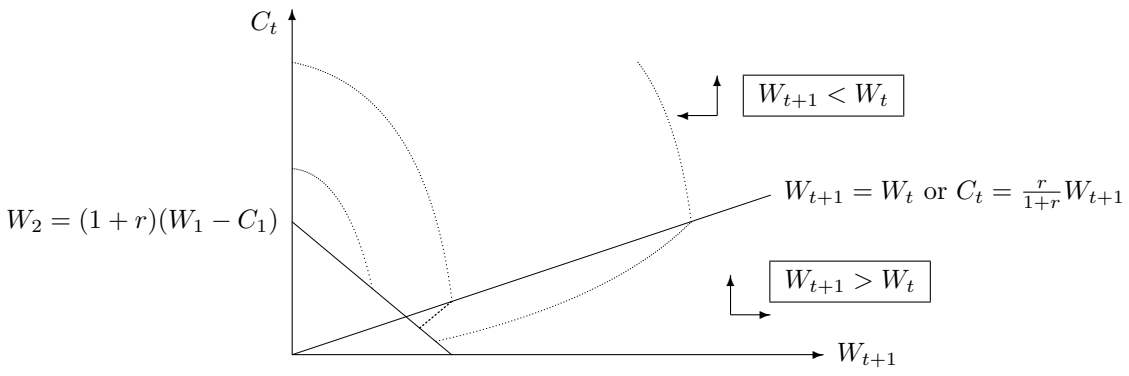
$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial C_t} &= \beta^{t-1} u'(C_t) - \lambda_t(1+r) = 0 \\ \frac{\partial \mathcal{L}}{\partial W_{t+1}} &= -\lambda_t + \lambda_{t+1}(1+r) = 0 \end{aligned}$$

Using $u'(C) = C^{-1/\sigma}$,

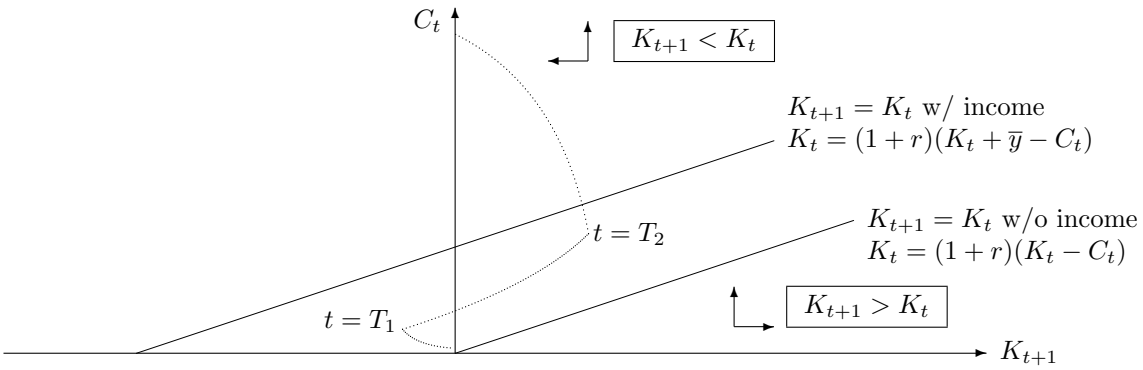
$$\begin{aligned} \frac{\beta^t u'(C_{t+1})}{\beta^{t-1} u'(C_t)} &= \frac{\lambda_{t+1}(1+r)}{\lambda_t(1+r)} \\ \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\sigma}} &= \frac{1}{1+r} \\ \frac{C_{t+1}}{C_t} &= [\beta(1+r)]^{\sigma} \end{aligned}$$

Define $g = [\beta(1+r)]^{\sigma} - 1$, which depends on β, r, and σ only.

(c) Note that C_t is increasing since $\beta(1+r) > 1$.



(d) In this special case, wealth-consumption phase diagram doesn't change. But if we draw capital-consumption phase diagram, it would be more complicated.



Until T_1 period, the agent has no income, and she has to borrow to consume. In these periods, she is above $K_{t+1} = K_t$ line, which means that her capital is decreasing. At T_1, she starts earning $\overline{y}$ each period. She is now below $K_{t+1} = K_t$ line, which means that her capital is increasing. If it is not the case, she cannot pay back her debt, because initial $K_1 = 0$.
After T_2, she retires and earns nothing, and thus she is again above $K_{t+1} = K_t$ line. Her consumption incearses and her capital decreases until T period. Possibly she can enter into capital decreasing phase before T_2 if T_2 is close enough to T.

(e) The end point changes so the optimal consumption path changes accordingly, but the shape of the path doesn't change much.

2000 Spring 2. 2 × 2 Economy

(a) A firm minimizes its cost.

$$\min_{K,L} wL + rK \qquad \text{subject to } q = K^{\alpha}L^{1-\alpha}$$

Lagrangian

$$\mathcal{L} = -wL - rK + \lambda(K^{\alpha}L^{1-\alpha} - q)$$

FOC's are

$$r = \lambda\alpha\left(\frac{L}{K}\right)^{1-\alpha},\ w = \lambda(1-\alpha)\left(\frac{K}{L}\right)^{\alpha} \text{ and } q = K^{\alpha}L^{1-\alpha}$$

The solution is

$$K^* = q\left(\frac{w\alpha}{r(1-\alpha)}\right)^{1-\alpha},\ L^* = q\left(\frac{r(1-\alpha)}{w\alpha}\right)^{\alpha}$$

Therefore, the cost function is

$$C(q,w,r) = wq\left(\frac{r(1-\alpha)}{w\alpha}\right)^{\alpha} + rq\left(\frac{w\alpha}{r(1-\alpha)}\right)^{1-\alpha} = qr^{\alpha}w^{1-\alpha}\left[\frac{1}{\alpha^{\alpha}} + \frac{1}{(1-\alpha)^{1-\alpha}}\right] = qr\left(\frac{w}{r}\right)^{\alpha} g(\alpha)$$

(b) Marginal rate of Technical substitution is

$$MRTS_{LK} = \frac{MP_L}{MP_K} = \frac{\alpha L^{\alpha-1}K^{1-\alpha}}{(1-\alpha)L^{\alpha}K^{-\alpha}} = \frac{\alpha K}{(1-\alpha)L}$$

At aggregate endowment $\left(\overline{L}, \overline{K}\right) = (100, 100)$,

$$MRTS^1_{LK} = \frac{\alpha_1}{1-\alpha_1} > \frac{\alpha_2}{1-\alpha_2} = MRTS^2_{LK}$$

So commodity 1 is more labor intensive. If q_1 increases, the labor would become more expensive, which is equivalent to that wage-rental ratio will increase. This follows from the Edgeworth box analysis where the optimal point moves far from the origin of commodity 1, so $MRTS$ increases according to the above inequality.

(c) From (a)

$$\frac{p_1}{p_2} = \frac{AC_1}{AC_2} = \left(\frac{w}{r}\right)^{\alpha_1-\alpha_2}\frac{g(\alpha_1)}{g(\alpha_2)}$$

(d) From (b)

$$\frac{w}{r} \in \left[MRTS^2_{LK}, MRTS^1_{LK}\right] = \left[\frac{\alpha_2}{1-\alpha_2}, \frac{\alpha_1}{1-\alpha_1}\right]$$

(e) Note that output price ratio and input price ratio have a positive relation. If output price ratio is large enough, this economy would specialize in commodity 1, and input price ratio would be equal to $MRTS^1_{LK}$. From (c), if

$$\frac{p_1}{p_2} \geq \left(\frac{\alpha_1}{1-\alpha_1}\right)^{\alpha_1-\alpha_2}\frac{g(\alpha_1)}{g(\alpha_2)}$$

or, equivalently

$$\frac{p_1}{p_2} \geq \left(\frac{\alpha_2}{\alpha_1}\right)^{\alpha_2}\left(\frac{1-\alpha_2}{1-\alpha_1}\right)^{1-\alpha_2}$$

then, the economy specializes in the productions of commodity 1.