2001 Spring 1. Life-cycle consumption

(a) A consumer's maximization problem is

$$\max \sum_{t=1}^{T} \delta^{t-1} \ln c_t \qquad \text{subject to} \quad A_{t+1} = (1+r)(A_t - c_t) \quad \forall t$$

Lagrangian is

$$\mathcal{L} = \sum_{t=1}^{T} \delta^{t-1} \ln c_t + \sum_{t=1}^{T} \lambda_t \left[(1+r)(A_t - c_t) - A_{t+1}\right]$$

FOC's are

$$\begin{aligned} \frac{\delta^{t-1}}{c_t} - \lambda_t(1+r) = 0 &\quad \text{w.r.t} \quad c_t \\ -\lambda_t + \lambda_{t+1}(1+r) = 0 &\quad \text{w.r.t} \quad A_{t+1} \\ (1+r)(A_t - c_t) - A_{t+1} = 0 &\quad \text{w.r.t} \quad \lambda_t \end{aligned}$$

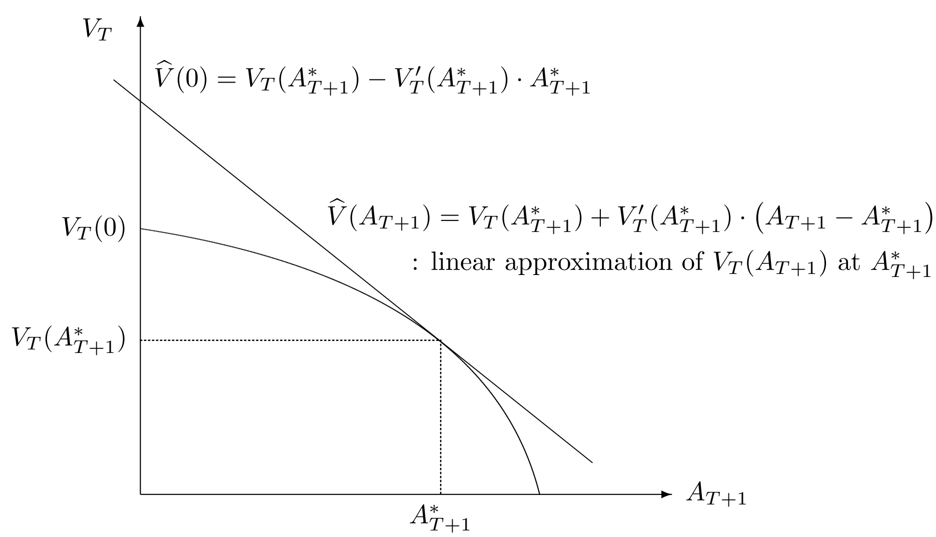
(b) Let (A^0_{t+1}, c^0_t) be the optimal choices under A^0_{T+1}, and (A^1_{t+1}, c^1_t) under A^1_{T+1}. It follows that

$$V(A^0_{T+1}) = \sum_{t=1}^{T} \delta^{t-1} \ln c^0_t \quad \text{and} \quad V(A^1_{T+1}) = \sum_{t=1}^{T} \delta^{t-1} \ln c^1_t$$

Since the feasible set $\{(A_{t+1}, c_t) | A_{t+1} \leq (1+r)(A_t - c_t) \forall t\}$ is convex, its convex combination is always feasible. Noting that V is the maximum of values attainable with feasible choices,

$$\begin{aligned} V\left([1-\lambda]A^0_{T+1} + \lambda A^1_{T+1}\right) &\geq \sum_{t=1}^{T} \delta^{t-1} \ln\left[(1-\lambda)c^0_t + \lambda c^1_t\right] \\ &> \sum_{t=1}^{T} \delta^{t-1}(1-\lambda) \ln c^0_t + \sum_{t=1}^{T} \delta^{t-1} \lambda \ln c^1_t \\ &= (1-\lambda)V(A^0_{T+1}) + \lambda V(A^1_{T+1}) \end{aligned}$$

which completes the proof, where the strict inequality comes from the fact that $\ln c$ is strictly concave.



The maximum of the consumer's lifetime utility is $V(0)$. Following Riley's suggestion in the class, if we know the optimal solution under A^*_{T+1}, we can find an upper bound for $V(0)$ in the following step. Since $V(A_{T+1})$ is concave,

$$V(A_{T+1}) \le V(A^*_{T+1}) + V'(A^*_{T+1})(A_{T+1} - A^*_{T+1})$$

or

$$V(A_{T+1}) \le V(A^*_{T+1}) + \lambda^*_T(A_{T+1} - A^*_{T+1})$$

by the Envelope theorem. Therefore,

$$V(0) \le V(A^*_{T+1}) - \lambda^*_T A^*_{T+1}$$

(c) Let $c_t = (1-\delta)A_t$, then

$$A_{t+1} = (1+r)(A_t - c_t) = (1+r)\delta A_t = (1+r)^t \delta^t A_1 > 0$$

Clearly, $c_t = (1-\delta)A_t > 0$, therefore, (A_{t+1}, c_t) satisfying the above condition is feasible.

(d) By the theorem, if the transversality condition holds, and a feasible (A_{t+1}, c_t) satisfies FOC, this choice is the optimal solution to the infinite horizon problem. FOC reads as

$$\frac{c_{t+1}}{c_t} = \delta(1+r)$$

A suggested choice given in (c) satisfies

$$c_{t+1} = (1-\delta)A_{t+1} = (1-\delta)(1+r)\delta A_t = (1+r)\delta c_t$$

Thus FOC is satisfied. Let's verify the transversality condition. From FOC,

$$\lambda_T = \frac{1}{1+r}\lambda_{T-1} = \cdots \frac{1}{(1+r)^{T-1}}\lambda_1 = \frac{1}{(1+r)^T c_1} = \frac{1}{(1+r)^T(1-\delta)A_1}$$

Since $A_{T+1} = (1+r)\delta A_T = \cdots = (1+r)^T\delta^T A_1$,

$$\lim_{T\to\infty} \lambda_T A_{T+1} = \lim_{T\to\infty} \frac{\delta^T}{1-\delta} = 0$$

The transversality condition also holds, therefore, the consumption path given in (c) is the optimal solution to the infinite horizon problem.

Actually, using AC and FOC,

$$\begin{aligned}
A_{T+1} &= (1+r)(A_T - c_T) \\
&= (1+r)^2(A_{T-1} - c_{T-1}) - (1+r)c_T \\
&\vdots \\
&= (1+r)^T A_1 - (1+r)^T c_1 - (1+r)^{T-1}c_2 - \cdots - (1+r)c_T \\
&= (1+r)^T A_1 - (1+r)^T c_1 - (1+r)^T\delta c_1 - \cdots - (1+r)^T\delta^{T-1}c_1 \\
&= (1+r)^T A_1 - (1+r)^T \frac{c_1(1-\delta^T)}{1-\delta}
\end{aligned}$$

Letting $A_{T+1} = 0$, we have

$$c_1 = \frac{1-\delta}{1-\delta^T}A_1$$

and also letting $T \to \infty$,

$$c_1 = (1-\delta)A_1$$

Therefore

$$c_t = (1+r)^{t-1}\delta^{t-1}(1-\delta)A_1$$

Taking similar iteration we did in the above,

$$A_{t+1} = (1+r)^t A_1 - (1+r)^t \frac{c_1(1-\delta^t)}{1-\delta} = (1+r)^t \delta^t A_1$$

thus we have the optimal solution (A_{t+1}, c_t) satisfying

$$c_t = (1-\delta)A_t$$

(e) (i) For the infinite horizon problem, c_1 will increase by only the portion $1-\delta$ of increase in A_1.

(ii) For the finite horizon problem also, c_1 will increase more than for the infinite case. To see this, note that

$$c_1 = \frac{1-\delta}{1-\delta^T} A_1$$

so the smaller T is, increase in A_1 raises the first period consumption more.

2001 Spring 2. Joint costs and fixed costs

(a) Lagrangian is

$$\mathcal{L} = q_1(130 - q_1) + q_2(120 - 2q_2) + q_3(200 - q_3) - 60L - 30q_1 - 20q_2 - 40q_3 + \lambda_1(L - q_1) + \lambda_2(L - q_2) + \lambda_3(L - q_3)$$

FOC's are

$$\begin{aligned}
130 - 2q_1 - 30 - \lambda_1 \leq 0 &\quad \text{WEI} \quad q_1^* > 0 \\
120 - 4q_2 - 20 - \lambda_2 \leq 0 &\quad \text{WEI} \quad q_2^* > 0 \\
200 - 2q_3 - 40 - \lambda_3 \leq 0 &\quad \text{WEI} \quad q_3^* > 0 \\
\lambda_1 + \lambda_2 + \lambda_3 - 60 \leq 0 &\quad \text{WEI} \quad Ł^* > 0 \\
L - q_1 \geq 0 &\quad \text{WEI} \quad \lambda_1^* > 0 \\
L - q_2 \geq 0 &\quad \text{WEI} \quad \lambda_2^* > 0 \\
L - q_3 \geq 0 &\quad \text{WEI} \quad \lambda_3^* > 0
\end{aligned}$$

Assuming $L = q_1 = q_2 = q_3$, we have $L^* = q_1^* = q_2^* = q_3^* = 37.5$, which implies $\lambda_2^* = -50 < 0$, a contradiction. Let $\lambda_2^* = 0$ and $L = q_1 = q_3$, then, $L^* = q_1^* = q_3^* = 50$, and $q_2^* = 25$. Since $\lambda_1^* = 0, \lambda_3^* = 60$, this is an optimal solution.

(b) With fixed costs, FOC's do not change, thus the optimal solution is still the same with that in (a). The aggregat profit is

$$50 \times 80 + 25 \times 70 + 50 \times 150 - 60 \times 50 - 30 \times 50 - 20 \times 25 - 40 \times 50 - F_1 - F_2 - F_3 = 1750$$

so it is better to produce. Now we can check for individual products.

$$\begin{aligned}
\text{product 1}: &\quad 50 \times 80 - 30 \times 50 - F_1 = 1500 \\
\text{product 2}: &\quad 25 \times 70 - 20 \times 25 - F_2 = -250 \\
\text{product 3}: &\quad 50 \times 150 - 40 \times 50 - F_3 = 3500
\end{aligned}$$

So, it would be better not to run facility 2, then the profit would be $1500 + 3500 - 30 \times 50 = 2000$. (We can include in the constraints, the condition that a profit from each product should be nonnegative, then we would have 6 constraints and corresponding Lagrange mulitpliers.)

(c) If this company chooses the same outputs with (b), the profit is 500, still positive. Also the profit from product 3 is 2000 given the fixed costs and $L^* = 50$, therefore the company would produce products 1 and 3, but not 2. If F_1 increases to 3500 while $F_3 = 2000$, then the company would produce only product 3, to make a profit of 500.

(d) The only condition that changes is 4th condition, which would be $\lambda_1 + \lambda_2 + \lambda_3 - 35 - 0.5L \leq 0$ Actually, the solution does not change, since $L* = 50$, $\lambda_1^* = \lambda_2^* = 0$, and $\lambda_3^* = 60$ still satisfy this condition at equality. Since Lagrangian function is concave, there is only one optimal solution, this is the only one.