2002 Spring 1. Short Takes

(a) It depends on whether the input is normal or inferior.

$$\frac{\partial MC}{\partial r_j} = \frac{\partial}{\partial r_j}\left(\frac{\partial C}{\partial q}\right) = \frac{\partial}{\partial q}\left(\frac{\partial C}{\partial r_j}\right) = \frac{\partial z_j}{\partial q}$$

where the last equality comes from Envelope theorem by differentiating $C(q, r) = \min\{r \cdot z|(q, z) \in Y\}$ with respect to r_j. If the input z_j is normal, this expression is positive, which implies that decrease in r_j shifts MC down. So both firms would increase output, and this cuts down market price from the demand function. If the input z_j is inferior, the above expression is negative, so the reasoning works in the exactly opposite direction.

(b) The statement is true.
The only thing we have to do is to introduce relevant markets and prices. For choice over time, we can introduce futures markets where we can trade a claim for future commodities. For choice under uncertainity, we can introduce contingent markets where we can trade for contingent commodities. Then analysis of equilibrium is the same for ordinary cases.

(c) The statement is true.
When all individuals have the same homothetic preferences, every agent consumes a fraction of aggregate endowment (which has to be proven below), so we can introduce a representative agent with aggregate endowment, and easily obtain equilibrium prices and allocations. To see that the claim is true, note first that

$$U(c) = U(c') \Longleftrightarrow U(\theta c) = U(\theta c') \quad \forall \theta > 0 \tag{1}$$

Define $c(p, I)$ as an optimal consumption under price p and income I. Since $c(p, I)$ is feasible under I, $\theta c(p, I)$ is feasible under θI. By definition,

$$U(\theta c(p, I)) \le U(c(p, \theta I)) \tag{2}$$

Again, $\frac{1}{\theta}c(p, \theta I)$ is feasible under I, so

$$U\left(\frac{1}{\theta}c(p, \theta I)\right) \le U(c(p, I))$$

By (1) homotheticity,

$$U(c(p, \theta I)) \le U(\theta c(p, I)) \tag{3}$$

Thus (2) and (3) together imply $U(c(p, \theta I)) = U(\theta c(p, I))$, or $c(p, \theta I) = \theta c(p, I)$. Let agent i has an endowment ω_i and define

$$\alpha_i = \frac{p \cdot \omega_i}{p \cdot \omega}$$

where $\omega = \sum_{i=1}^n \omega_i$ is aggregate endowment. Then i's consumption is

$$c(p, p \cdot \omega_i) = c(p, \alpha_i p \cdot \omega) = \alpha_i c(p, p \cdot \omega)$$

By market clearing condition, $\sum_{i=1}^n \alpha_i c(p, p \cdot \omega) = \omega$, which is equivalent to $c(p, p \cdot \omega) = \omega$, and thus

$$c(p, p \cdot \omega_i) = \alpha_i \omega$$

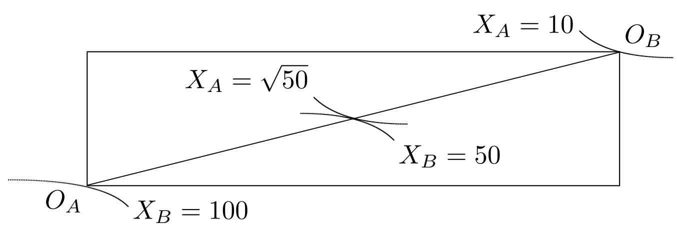
Therefore, every agent consumes a fraction of aggregate endowment ω.

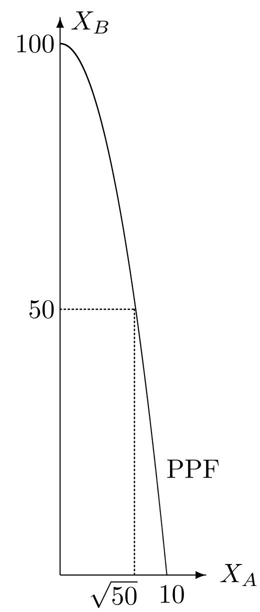
2002 Spring 4. 2 × 2 Economy

(a) Since $MRTS$ of two commodities A and B are

$$\begin{aligned} MRTS_A &= \frac{\frac{1}{4}L_A^{-\frac{3}{4}}K_A^{\frac{1}{4}}}{\frac{1}{4}L_A^{\frac{1}{4}}K_A^{-\frac{3}{4}}} = \frac{K_A}{L_A} \\ MRTS_B &= \frac{\frac{1}{2}L_B^{-\frac{1}{2}}K_B^{\frac{1}{2}}}{\frac{1}{2}L_B^{\frac{1}{2}}K_A^{-\frac{1}{2}}} = \frac{K_B}{L_B} \end{aligned}$$

So $MRTS_A|_{(200,50)} = MRTS_B|_{(200,50)}$ and the optimal production path is a diagonal in the Edgeworth box. Accordingly, the production possibility frontier looks like below.





(b) Actually we can derive a equation of PPF. The optimal production path shows that $L_A = 4K_A$ and $L_B = 4K_B$. Plugging in these into production functions gives

$$x_A = L_A^{\frac{1}{4}}K_A^{\frac{1}{4}} = (2K_A)^{\frac{1}{2}} \tag{4}$$

and

$$x_B = L_B^{\frac{1}{2}}K_B^{\frac{1}{2}} = 2K_B = 2(50 - K_A) = 100 - 2K_A = 100 - x_A^2$$

where the third equation comes from $K_A + K_B = 50$, and the last equation comes from (4). So $x_A^2 + x_B = 100$ is the equation of PPF. Obtain the derivatives of PPF at the endpoints as follows.

$$\left.\frac{\partial x_B}{\partial x_A}\right|_{x_A=10,x_B=0} = -2x_A\Big|_{x_A=10,x_B=0} = -20$$

and

$$\left.\frac{\partial x_B}{\partial x_A}\right|_{x_A=0,x_B=100} = -2x_A\Big|_{x_A=0,x_B=100} = 0$$

When outputs are freely traded, we would try to maximize revenue in order to buy more from the revenue. If the price ratio is greater than the absolute value of the slope of PPF at (10,0), or equivalently,

$$\frac{P_A}{P_B} \geq 20$$

then the economy would specialize in the production of commodity A. But the economy will never specialize in the production of commodity B, since the slope of PPF is 0 at (0,100).

(c) Now PPF is a budget set. So the utility maximization problem is

$$\max 2\log x_A + 2\log x_B \quad \text{subject to } x_A^2 + x_B = 100$$

Set up the Lagrangian as

$$\mathcal{L} = 2\log x_A + 2\log x_B + \lambda\left(100 - x_A^2 - x_B\right)$$

FOC writes as

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_A} &= \frac{2}{x_A} - 2\lambda x_A = 0 \\
\frac{\partial \mathcal{L}}{\partial x_B} &= \frac{2}{x_B} - \lambda = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= 100 - x_A^2 - x_B = 0
\end{aligned}$$

The solution would be

$$\frac{1}{\lambda} = \frac{100}{3} \quad , \quad x_A = \frac{10}{\sqrt{3}} \quad , \quad x_B = \frac{200}{3}$$

and the equilibrium price is equal to the absolute value of the slope of PPF at $\left(\frac{10}{\sqrt{3}}, \frac{200}{3}\right)$, which is

$$\frac{P_A}{P_B} = -\left.\frac{\partial x_B}{\partial x_A}\right|_{x_A=\frac{10}{\sqrt{3}}, x_B=\frac{200}{3}} = \frac{20}{\sqrt{3}}$$