2003 Spring 1. Life Cycle Saving

(a) Consumer's optimization problem is

$$\max_{c_1,c_2,c_3} u(c_1)+\delta u(c_2)+\delta^2 u(c_3)$$
$$\text{subject to } c_1+\frac{c_2}{1+r}+\frac{c_3}{(1+r)^2}=W$$

Lagrangian

$$\mathcal{L}=\sqrt{c_1}+\delta\sqrt{c_2}+\delta^2\sqrt{c_3}+\lambda\left(W-c_1-\frac{c_2}{1+r}-\frac{c_3}{(1+r)^2}\right)$$

FOC's are

$$\begin{aligned}
\frac{1}{2\sqrt{c_1}} &= \lambda\\
\frac{\delta}{2\sqrt{c_2}} &= \frac{\lambda}{1+r}\\
\frac{\delta^2}{2\sqrt{c_3}} &= \frac{\lambda}{(1+r)^2}\\
W &= c_1+\frac{c_2}{1+r}+\frac{c_3}{(1+r)^2}
\end{aligned}$$

(b) From FOC's

$$\frac{1}{2\sqrt{c_1}}=\frac{\delta(1+r)}{2\sqrt{c_2}}=\frac{\delta^2(1+r)^2}{2\sqrt{c_3}}$$

Taking squares and multiplying by 4,

$$\frac{1}{c_1}=\frac{\delta^2(1+r)^2}{c_2}=\frac{\delta^4(1+r)^4}{c_3}$$

Defining $\gamma=\delta^2(1+r)$, rewrite this as

$$\frac{1}{c_1}=\frac{\gamma(1+r)}{c_2}=\frac{\gamma^2(1+r)^2}{c_3}$$

and the desired equation follows.

(c) Combining (b) and the resource constraint,

$$W=c_1+\gamma c_1+\gamma^2 c_1$$

so

$$c_1^*=\frac{1-\gamma}{1-\gamma^3}W$$

(d) We can obtain FOC similar to (b) in T periods case. So the solution is very similar to (c) as well. Indeed

$$\frac{c_t}{(1+r)^{t-1}}=c_1\gamma^{t-1}$$

Combining this with the resource constraint, we have

$$W=c_1(1+\cdots+\gamma^{T-1})$$

so

$$c_1^* = \frac{1-\gamma}{1-\gamma^T} W$$

(e) If the wage in period 1 increases by 1, W increases by 1, so c_1^* increases by

$$\frac{\partial c_1^*}{\partial W} = \frac{1-\gamma}{1-\gamma^T}$$

If T is large and γ is less than 1, this is close to $1-\gamma$. If γ is greater than 1, this is close to 0. When the wage in period T increases by 1, W increases by $1/(1+r)^{T-1}$, so c_1^* increases by

$$\frac{1}{(1+r)^{T-1}} \cdot \frac{\partial c_1^*}{\partial W} = \frac{1}{(1+r)^{T-1}} \cdot \frac{1-\gamma}{1-\gamma^T}$$

If T is large, $1/(1+r)^{T-1}$ is close to 0, so the marginal propensity to consume is also close to 0.

(f) If $\gamma = 0.9$, as T goes to inifinity, c_1^* converges to $(1-\gamma)W = 0.1W$. But if $\gamma = 1.1$, as T goes to infinity, c_1^* converges to 0. Note that this solution is the only solution that satisfies FOC's and the resource constraint, but it is not utility maximizing solution.

2003 Spring 2. Time and Uncertainty

(a) The aggregate endowment is $\omega = (200, 200, 300)$. The equilibrium state price is

$$p = \frac{\partial U}{\partial c}\bigg|_{\omega} = \left(\frac{1}{c_1}, \frac{\pi_1}{c_{21}}, \frac{\pi_2}{c_{22}}\right)\bigg|_{\omega} = \left(\frac{1}{200}, \frac{\pi_1}{200}, \frac{\pi_2}{300}\right) \quad \text{or} \quad p = \left(1, \pi_1, \frac{2}{3}\pi_2\right)$$

(b) Asset prices are

$$p_a = 100 \times \frac{1}{200} + 100 \times \frac{\pi_1}{200} + 100 \times \frac{\pi_2}{300} = \frac{1}{2} + \frac{1}{2}\pi_1 + \frac{1}{3}\pi_2$$
$$p_b = 100 \times \frac{1}{200} + 100 \times \frac{\pi_1}{200} + 200 \times \frac{\pi_2}{300} = \frac{1}{2} + \frac{1}{2}\pi_1 + \frac{2}{3}\pi_2$$

(c) Since utility function is homothetic and identical, all individuals comsume a fraction of the aggregate endowment. Therefore, they can replicate optimal state claims by trading assets, even though there are 3 states and only 2 assets. Prices and consumption do not change. We can show this. The utility function in terms of assets is

$$u(z_a, z_b) = \ln(100z_a + 100z_b) + \pi_1 \ln(100z_a + 100z_b) + \pi_2 \ln(100z_a + 200z_b)$$

This is homothetic and identical. The representative agent has $(z_a, z_b) = (1, 1)$, so the equilibrium asset prices are

$$p_a = \frac{\partial u}{\partial z_a}\bigg|_{(1,1)} = \left(\frac{100}{100z_a + 100z_b} + \frac{100\pi_1}{100z_a + 100z_b} + \frac{100\pi_2}{100z_a + 200z_b}\right)\bigg|_{(1,1)} = \frac{1}{2} + \frac{1}{2}\pi_1 + \frac{1}{3}\pi_2$$
$$p_a = \frac{\partial u}{\partial z_b}\bigg|_{(1,1)} = \left(\frac{100}{100z_a + 100z_b} + \frac{100\pi_1}{100z_a + 100z_b} + \frac{200\pi_2}{100z_a + 200z_b}\right)\bigg|_{(1,1)} = \frac{1}{2} + \frac{1}{2}\pi_1 + \frac{2}{3}\pi_2$$

(d) They will be equally well off compared with the outcome with the outcome when all state claims can be traded for the same reason with in (c). Asset prices will be

$$p_a = \frac{1}{2} + \frac{1}{2}\pi_1 + \frac{1}{3}\pi_2 + \frac{1}{4}\pi_3$$

$$p_b = \frac{1}{2} + \frac{1}{2}\pi_1 + \frac{2}{3}\pi_3 + \frac{3}{4}\pi_3$$