

2004 Fall 1. Walrasian Equilibrium

(a) Let x_{tj}^h be consumer h 's consumption of commodity j in period t . Each consumer has an endowment $(\omega_{11}^h, \omega_{12}^h, 0, 0, \dots, 0)$, and the aggregate endowment is $(\omega_{11}, \omega_{12}, 0, 0, \dots, 0)$. Each has also a utility function

$$u^h(x^h) = U^h(x_{11}^h, x_{12}^h) + \delta \sum_{s=1}^S \pi_s U^h(x_{21}^h, x_{22s}^h)$$

where π_s is the probability that state s occurs, and $\sum_{s=1}^S \pi_s = 1$. Each has a share s^{hf} of the firm f . Firm f produces $y_{21} = -y_{11}^f$ and $y_{22} = F(-y_{12}^f)$, S -valued function, where y_{11}^f and y_{12}^f are (negative) purchases of inputs.

(b) There are $S + 3$ contingent commodities, so we need $S + 2$ relative prices.

(c) Every firm maximizes its profit under p , thus

$$p_{11} = p_{21} \quad , \quad p_{12} = \sum_{s=1}^S p_{22s} \frac{\partial F(-y_{12}^f)}{\partial y_{22s}^f}$$

Every consumer maximizes one's utility under p , thus

$$p_1 = \left. \frac{\partial U^h}{\partial x^h} \right|_{(x_{11}^h, x_{12}^h)} \quad , \quad p_{21} = \delta \sum_{s=1}^S \pi_s \left. \frac{\partial U^h}{\partial x_{21}^h} \right|_{(x_{21}^h, x_{22s}^h)} \quad \text{and} \quad p_{22s} = \delta \pi_s \left. \frac{\partial U^h}{\partial x_{22s}^h} \right|_{(x_{21}^h, x_{22s}^h)} \quad \text{for all } h$$

All consumers should satisfy their own budget constraint.

$$p \cdot (\omega^h - x^h) + \sum_{f=1}^F s^{hf} \pi^f = 0$$

where $\pi^f = p \cdot y^f$ is firm f 's profit. Market clearing conditions are

$$p \cdot (\omega + y - x) = 0 \quad \text{with} \quad \omega + y - x \geq 0$$

(d) If there are futures markets, these are Walrasian equilibrium prices. If there is no futures market, we need one bond market (or similar) and bond price for these to be Walrasian equilibrium prices. If there is a bond market, the optimal storage of commodity 1 and the optimal production of commodity 2 will be decided in the market, and all agents will intertemporally optimize their own consumption by trading in the bond market. Intertemporal equilibrium interest rate will be 0.

(e) If all consumers and firms correctly know there will be no endowment of commodity 1 in period 2, they will choose optimal storage of commodity 1 as a whole economy, so that spot prices will be consistent with implicit futures price and interest rate at the complete market equilibrium. This would yield the same maximization problem for consumers and firms and the same equilibrium conditions, thus the Walrasian equilibrium of this economy would be the same with the complete market equilibrium.

(f) We have at least two relative prices, one in spot market, and the other in future spot market. If the number of shares is greater than or equal to S , we can replicate the complete market equilibrium outcome. If the number of shares is less than S , we can replicate also under some conditions. The conditions are that all consumers have the same homothetic utility function, and that the aggregate endowment can be replicated by a linear combination of shares in a stock. If this condition is satisfied, all consumers will choose a fraction of the aggregate endowment.

2004 Fall 2. Cobb-Dougals Economy

(a) A firm minimizes its cost.

$$\min_{K,L} wL + rK \quad \text{subject to } x = K^\alpha L^{1-\alpha}$$

Lagrangian

$$\mathcal{L} = -wL - rK + \lambda(K^\alpha L^{1-\alpha} - x)$$

FOC's are

$$r = \lambda\alpha \left(\frac{L}{K}\right)^{1-\alpha}, \quad w = \lambda(1-\alpha) \left(\frac{K}{L}\right)^\alpha \quad \text{and} \quad x = K^\alpha L^{1-\alpha}$$

The solution is

$$K^* = x \left(\frac{w\alpha}{r(1-\alpha)}\right)^{1-\alpha}, \quad L^* = x \left(\frac{r(1-\alpha)}{w\alpha}\right)^\alpha$$

Therefore, the cost function is

$$C(x, w, r) = wx \left(\frac{r(1-\alpha)}{w\alpha}\right)^\alpha + rx \left(\frac{w\alpha}{r(1-\alpha)}\right)^{1-\alpha} = x \left(\frac{r}{\alpha}\right)^\alpha \left(\frac{w}{1-\alpha}\right)^{1-\alpha}$$

(b) We allocate capital and labor of 100 units into production of two commodities to maximize utility. Production functions and utility function are given, so we can substitute production functions into utility function, and maximize utility with respect to capital and labor.

(c) $x_1 = K_1^{2/3} L_1^{1/3}$ and $x_2 = K_1^{1/3} L_1^{2/3}$, utility can be expressed as

$$U = 2 \ln K_1^{2/3} L_1^{1/3} + \ln(100 - K_1)^{1/3} (100 - L_1)^{2/3}$$

FOC's are

$$2 \cdot \frac{1}{3} \cdot \frac{K_1^{2/3} L_1^{-2/3}}{K_1^{2/3} L_1^{1/3}} = \frac{2}{3} \cdot \frac{(100 - K_1)^{1/3} (100 - L_1)^{-1/3}}{(100 - K_1)^{1/3} (100 - L_1)^{2/3}}$$

$$2 \cdot \frac{2}{3} \cdot \frac{K_1^{-1/3} L_1^{1/3}}{K_1^{2/3} L_1^{1/3}} = \frac{1}{3} \cdot \frac{(100 - K_1)^{-2/3} (100 - L_1)^{2/3}}{(100 - K_1)^{1/3} (100 - L_1)^{2/3}}$$

Thus $L_1 = 50, L_2 = 50, K_1 = 80, K_2 = 20$. Equilibrium input price ratio is

$$\frac{w}{r} = \frac{MP_{L1}}{MP_{K1}} \Big|_{(50,80)} = \frac{MP_{L2}}{MP_{K2}} \Big|_{(50,20)} = \frac{1}{2} \cdot \frac{K}{L} \Big|_{(50,80)} = \frac{4}{5}$$

Since both production functions are constant returns to scale, we can introduce firms which maximizes their profit to be 0.

$$p_1 = AC_1 = r^{2/3} w^{1/3} 3^{1/3} \left(\frac{3}{2}\right)^{2/3}, \quad p_2 = AC_2 = r^{1/3} w^{2/3} 3^{1/3} \left(\frac{3}{2}\right)^{2/3}$$

Thus

$$\frac{p_1}{p_2} = \left(\frac{w}{r}\right)^{-1/3} = \left(\frac{5}{4}\right)^{1/3}$$

(d) When utility function is not given, we cannot characterize an input price ratio exactly, but we can obtain a range of input price ratio from Edgeworth box of production. It would be

$$\frac{w}{r} \in \left[\frac{MP_{L1}}{MP_{K1}} \Big|_{(100,100)}, \frac{MP_{L2}}{MP_{K2}} \Big|_{(100,100)} \right] = \left[\frac{1}{2}, 2 \right]$$