2004 Spring 1. Homothetic Preferences

(a) Homothetic utility function is a utility function u that satisfies

$$u(x) \geq u(y) \Leftrightarrow u(kx) \geq u(ky) \quad \text{for all } k > 0$$

Under these preferences, the income expansion path will be a ray from the origin. To see this, let x^*_M be the optimal consumption under income M. Since a budget constraint is linear, both x^*_M and $\frac{1}{k}x^*_{kM}$ are feasible under M. By definition of x^*,

$$u\left(x^*_M\right) \geq u\left(\frac{1}{k}x^*_{kM}\right)$$

By homotheticity,

$$u\left(kx^*_M\right) \geq u\left(x^*_{kM}\right)$$

Also both x^*_{kM} and kx^*_M are feasible under kM. Again by definition of x^*,

$$u\left(x^*_{kM}\right) \geq u\left(kx^*_M\right)$$

So we have

$$u\left(x^*_{kM}\right) = u\left(kx^*_M\right)$$

It follows that[1]

$$x^*_{kM} = kx^*_M$$

Differentiating this with respect to k,

$$\frac{\partial x^*_{kM}}{\partial M} \cdot M = x^*_M$$

Letting $k = 1$, we have

$$E(x^*, M) = \frac{\partial x^*}{\partial M} \cdot \frac{M}{x^*} = 1$$

(b) If a utility function is identical as well as homothetic, both consumers will consume a fraction of the aggregate endowment, so the equilibrium price p will satisfy

$$p = \left.\frac{\partial U^1}{\partial x}\right|_{x^*_1} = \left.\frac{\partial U^2}{\partial x}\right|_{x^*_2} = \left.\frac{\partial U}{\partial x}\right|_{(\omega_1,\omega_2)}$$

The last equality follows from homotheticity of the utility function. If a utility function is not identical, we cannot say about characteristic of the equilibrium price. It depends on how much each has and how one's utility function is different from the other's.

(c) Given a price p_1, comsumer 2 will maximize her utility, so her consumption vector is a function of p_1. This is called "offer curve". Consumer 1's budget set is the offer curve. He will maximize his utility over the offer curve. Assume that $p_2 = 1$ and consumer 2 owns $(0, \omega^2_2)$. With Cobb-Doublas utility function $u(x) = x_1^\alpha x_2^\beta$, if consumer 1 offers p_1, consumer 2 will choose to consume

$$x^2_1 = \frac{1}{p_1}\frac{\alpha}{\alpha+\beta}\omega^2_2 \quad , \quad x^2_2 = \frac{\beta}{\alpha+\beta}\omega^2_2$$

Since consumer 1 can consume all that are left, he will offer $p_1 = \infty$ to consume $\left(\omega^1_1, \omega^1_2 + \frac{\alpha}{\alpha+\beta}\omega^2_2\right)$.

(d) False. This is true only in the weak sense. Note that consumer 1 can suggest the Walrasian equilibrium price. So he cannot be worse off when he can choose any price he wants. But consider, for example, the case in which consumer 2 has lexicographic preferences which put priority on commodity 2. Then consumer 2 would not sell any of her commodity 2 even if consumer 1 offers very low p_1. So consumer 1 is in this case as well off as in the Walrasian equilibrium.

[1] This is true even if x^* is multi-valued function as long as u is strictly increasing.

2004 Spring 2. Robinson Crusoe economy with uncertainty

(a) Suppose x^* is strictly positive. The maximization problem is

$$\max_x \ln(\omega_1 - x) + \pi_1 \ln(\omega_2 + \alpha x) + \pi_2 \ln(\omega_2 + \alpha x)$$

FOC is

$$\frac{1}{\omega_1 - x^*} = \frac{\alpha\pi_1}{\omega_2 + \alpha x^*} + \frac{\alpha\pi_2}{\omega_2 + \alpha x^*}$$

Since $\pi_1 + \pi_2 = 1$, this simplifies as

$$x^* = \frac{\alpha\omega_1 - \omega_2}{2\alpha}$$

Therefore x^* is strictly positive when the RHS is strictly positive, or equivalently, $\alpha\omega_1 > \omega_2$.

(b) The maxmization problem is

$$\max_{x,y} \ln(\omega_1 - x - y) + \pi_1 \ln(\omega_2 + \alpha x + \beta y) + \pi_2 \ln(\omega_2 + \alpha x)$$

Since x^* is strictly positive, FOC's are

$$\begin{aligned} \frac{1}{\omega_1 - x^* - y^*} &= \frac{\alpha\pi_1}{\omega_2 + \alpha x^* + \beta y^*} + \frac{\alpha\pi_2}{\omega_2 + \alpha x^*} \\ \frac{1}{\omega_1 - x^* - y^*} &\geq \frac{\beta\pi_1}{\omega_2 + \alpha x^* + \beta y^*} \quad \text{with eq if } y^* > 0 \end{aligned}$$

Combining two conditions,

$$\frac{\alpha\pi_1}{\omega_2 + \alpha x^* + \beta y^*} + \frac{\alpha\pi_2}{\omega_2 + \alpha x^*} \geq \frac{\beta\pi_1}{\omega_2 + \alpha x^* + \beta y^*} \quad \text{with eq if } y^* > 0$$

If strict inequality holds even when $y^* = 0$, then optimal is $y^* = 0$. This condition is $\alpha > \beta\pi_1$.

(c) These conclusions do not include any risk aversion parameter. (a) is because production of no risky asset does not require consideration of risk aversion. (b) is because, when risky asset is not produced at all, there is no difference in marginal rate of substitution (or marginal utility of risky asset) between risk averse agent and risk neutral agent.[2]

(d) The optimal price will satisfy $p_1^* = \alpha p_{21}^* + \alpha p_{22}^*$ and $p_1^* = \beta p_{21}^*$ since otherwise firms can always be better off by changing their output. Since $p_1^* = 1$,

$$p_{21}^* = \frac{1}{\beta} \quad \text{and} \quad p_{22}^* = \frac{1 - \alpha\beta}{\alpha}$$

(e) Continue to assume strict positive x^* and y^*. Before firms purchase inputs, the market value of each firm is nothing. After they purchase inputs, they can produce, so the market value is

$$p_A = \max\{p_1, \alpha p_{21} + \alpha p_{22}\}x \quad \text{and} \quad p_B = \max\{p_1, \beta p_2\}y$$

At the equilibrium,

$$p_A^* = p_1 x^* \quad \text{and} \quad p_B^* = p_1 y^*$$

[2] My answer is different from Venky's. Venky said that if the utility function is linear, then the conditions would be different from what we got in (a) and (b). I agree with him, but I am not sure as to how we should write down the answer.