2005 Fall 1. Production and cost

(a) $C(q,r) = \min\{r \cdot z | (q,z) \in Y\}$

(b) Let z^i be the cost minimizing input vector corresponding to r^i $(i = 0, 1, \lambda)$ where $r^\lambda = \lambda r^1 + (1-\lambda)r^0$. By definition,

$$C(q, r^1) = r^1 \cdot z^1 \leq r^1 \cdot z^\lambda \tag{1}$$
$$C(q, r^0) = r^0 \cdot z^0 \leq r^0 \cdot z^\lambda \tag{2}$$

Adding $\lambda\times(1)$ to $(1-\lambda)\times(2)$,

$$\lambda C(q, r^1) + (1-\lambda)C(q, r^0) \leq r^\lambda \cdot z^\lambda = C(q, r^\lambda)$$

which completes the proof.

(c) $C(q,r)$ is convex in q when Y is convex. Let z^i be the cost minimizing input vector corresponding to q^i $(i = 0, 1)$. Since Y is convex, $(q^\lambda, z^\lambda) \in Y$ where $q^\lambda = \lambda q^1 + (1-\lambda)q^0$ and $z^\lambda = \lambda z^1 + (1-\lambda)z^0$. By definition of cost function,

$$C(q^\lambda, r) \leq r \cdot z^\lambda = \lambda r \cdot z^1 + (1-\lambda) r \cdot z^0 = \lambda C(q^1, r) + (1-\lambda)C(q^0, r)$$

which completes the proof that $C(q,r)$ is convex in q.

(d) By the first law of supply, profit maximizing q_1 rises as the result of increase in p_1. However we cannot say about q_2 unless there are some more conditions. If the firm is a price take in input markets as well and all the other prices are unchanged, and if output 2 is independently produced, optimal q_2 does not change. (We can make so that two outputs are indenpendently produced and the production set is convex.) q_2 basically depends on the correlation of two outputs. If two outputs are positively correlated (one production stimulates the other), q_2 also rises when p_1 rises. If the correlation is negative, q_2 will decrease as the result of increase in p_1.

(e) A firm minimizes its cost under the constraint $\ln(1 + z_1 + z_2) \geq q$, equivalently, $z_1 + z_2 \geq e^q - 1$. Obviously, the way to minimize the cost is to use the cheaper input only. So the cost function is

$$C(q,r) = \min\{r_1, r_2\}(e^q - 1)$$

2005 Fall 2. Choice over Time

(a) The optimization problem is

$$\max_{\{c_t, K_{t+1}\}_{t=1}^T} \sum_{t=1}^{T} \delta^{t-1} u(c_t)$$
$$\text{subject to } K_{t+1} = (1+r)(K_t - c_t) \qquad t = 1, \cdots, T$$

Lagrangian

$$\mathcal{L} = \sum_{t=1}^{T} \{\delta^{t-1} u(c_t) + \lambda_t[(1+r)(K_t - c_t) - K_{t+1}]\}$$

FOC's are

$$\begin{aligned} \delta^{t-1} u'(c_t) &= \lambda_t \qquad t = 1, \cdots, T \quad (\text{w.r.t. } c_t) \\ \lambda_{t+1}(1+r) &= \lambda_t \qquad t = 1, \cdots, T \quad (\text{w.r.t. } K_{t+1}) \end{aligned}$$

From the second equation, we have

$$\frac{\lambda_{t+1}}{\lambda_t} = \frac{1}{1+r}$$

so by the first equation,

$$\frac{\delta u'(c_{t+1})}{u'(c_t)} = \frac{1}{1+r}$$

or

$$\delta \left(\frac{c_t}{c_{t+1}} \right)^{\frac{1}{\sigma}} = \frac{1}{1+r}$$

Rewrite this as

$$\frac{c_{t+1}}{c_t} = [(1+r)\delta]^{\sigma}$$

Define

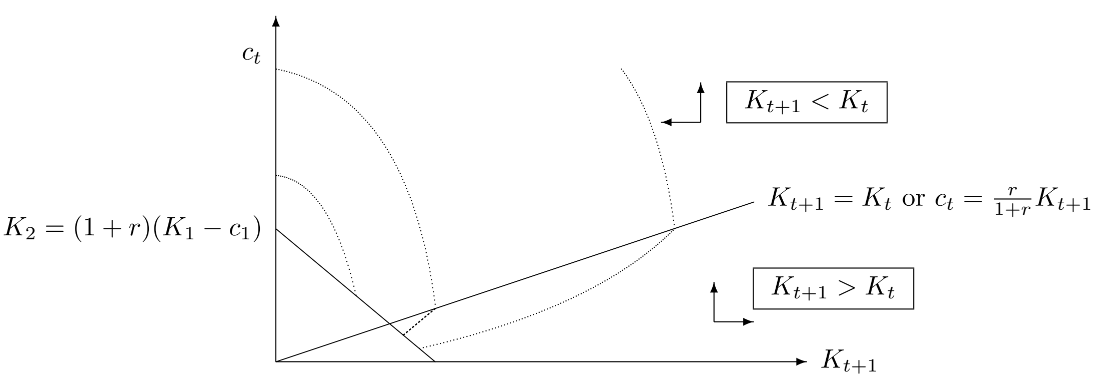
$$\theta = \frac{[(1+r)\delta]^{\sigma}}{1+r}$$

then

$$\frac{c_{t+1}}{c_t} = (1+r)\theta$$

(b) Note that $\theta < 1$ when $\sigma = 0$. By continuity, $\theta < 1$ if σ is close enough to 0. Since $(1+r)\delta > 1$, for sufficiently large σ, $[(1+r)\delta]^{\sigma} > 1 + r$.

(c) Note that c_t is increasing since $(1+r)\delta > 1$.



(d) From $K_{t+1} = (1+r)(K_t - c_t)$,

$$\frac{K_{t+1}}{1+r} = K_t - c_t$$

$$\frac{K_{t+1}}{(1+r)^2} = \frac{K_t}{1+r} - \frac{c_t}{1+r} = K_{t-1} - c_{t-1} - \frac{c_t}{1+r}$$

so

$$\frac{K_{t+1}}{(1+r)^t} = K_1 - c_1 - \frac{c_2}{1+r} - \cdots - \frac{c_t}{(1+r)^{t-1}}$$

Using (ii), rewrite this as

$$c_1 + c_1\theta + \cdots + c_1\theta^{t-1} + \frac{K_{t+1}}{(1+r)^t} = K_1$$

Let $t = T$. Assuming $K_{T+1} = 0$, we have

$$c_1\left(1 + \cdots + \theta^{T-1}\right) = K_1$$

or

$$c_1 = \frac{1-\theta}{1-\theta^T} K_1$$

(e) If $\theta < 1$, as T grows to infinity, c_1 converges to $(1-\theta)K_1$. If $\theta > 1$,

$$\begin{aligned} \lim_{T\to\infty} c_1 &= \lim_{T\to\infty} \frac{1-\theta}{1-\theta^T} K_1 \\ &= \lim_{T\to\infty} \frac{\left(\frac{1}{\theta}\right)^T - \left(\frac{1}{\theta}\right)^{T-1}}{\left(\frac{1}{\theta}\right)^T - 1} K_1 \\ &= 0 \end{aligned}$$

(f) When $\theta < 1$, the solution is all right, in the sense that it indeed maximizes intertemporal utility in infinite horizon problem. When $\theta > 1$, however, the solution, which is to consume 0 at every period in infinite case, deos not maximizes utility. Obviously, if one consumes K_1 in the first period, utility would be higher than when one consumes nothing in any period.[1] Actually, the transversality condition is not satisfied if $\theta > 1$.

[1] Of course, consuming all in the first period is not optimal. We can have more utility by allocating some endowment to later periods. In fact, there is no optimal comsumption stream in this case.