2005 Spring 5. A q-unit Auction

See Hong's TA note.
See Mike's TA note week 11.

2005 Spring 6. Bus Service

(a) For Pareto optimality, maximize social benefit

$$\max \sum_{i=1}^{n} v_i(f,l) - c(f,l)$$

FOC's are

$$\begin{aligned} \sum_{i=1}^{n} \frac{\partial}{\partial f} v_i(f,l) &= \frac{\partial}{\partial f} c(f,l) \\ \sum_{i=1}^{n} \frac{\partial}{\partial l} v_i(f,l) &= \frac{\partial}{\partial l} c(f,l) \end{aligned}$$

(b) For the solution of FOC to be optimal, it would be sufficient that the objective function is concave. So the sufficient condition is that v_i is concave for all i, and c is convex.

(c) Lindahl equilibrium is $[(f_i, l_i), f_0, l_0, (p_i), p_0]$ such that
(1) $v_i^*(p_i) = v_i(f_i, l_i) - p_i(f_i, l_i)$
(2) $v_0^*(p_0) = p_0(f_0, l_0) - c(f_0, l_0)$
(3) $f_i = f_0$ and $l_i = l_0$ for all i
(4) $\sum_{i=1}^{n} p_i = p_0$
Here, note that all prices are elements of R^2.

(d) By (1) and (2), and the definition of v^*, for any $[(f_i', l_i'), f_0', l_0']$

$$\sum_{i=1}^{n} [v_i(f_i, l_i) - p_i(f_i, l_i)] + [p_0(f_0, l_0) - c(f_0, l_0)] \geq \sum_{i=1}^{n} [v_i(f_i', l_i') - p_i(f_i', l_i')] + [p_0(f_0', l_0') - c(f_0', l_0')]$$

Paying attention to feasible allocation $[(f_i', l_i'), f_0', l_0']$ only, satisfying (3),

$$\sum_{i=1}^{n} v_i(f_i, l_i) - \sum_{i=1}^{n} p_i(f_0, l_0) + p_0(f_0, l_0) - c(f_0, l_0) \geq \sum_{i=1}^{n} v_i(f_i', l_i') - \sum_{i=1}^{n} p_i(f_0', l_0') + p_0(f_0', l_0') - c(f_0', l_0')$$

Appealing to (4),

$$\sum_{i=1}^{n} v_i(f_i, l_i) - c(f_0, l_0) \geq \sum_{i=1}^{n} v_i(f_i', l_i') - c(f_0', l_0')$$

This is true for all feasible allocation $[(f_i', l_i'), f_0', l_0']$. So LE allocation $[(f_i, l_i), f_0, l_0]$ is Pareto efficient.

(e) The poor maximizes

$$\max v_P(f,l) - \frac{1}{n} c(f,l)$$

FOC's are

$$\begin{aligned} 2f^{-\frac{1}{3}} - \frac{1}{n} \cdot 2f &= 0 \\ l^{-\frac{2}{3}} - \frac{1}{n} \cdot 2l &= 0 \end{aligned}$$

The solution is

$$f_P = n^{\frac{3}{4}} \quad , \quad l_P = \left(\frac{n}{2}\right)^{\frac{3}{5}}$$

and per capita payment would be

$$t_P = \frac{1}{n}(f_P^2 + l_P^2) = \frac{1}{n}\left[n^{\frac{3}{2}} + \left(\frac{n}{2}\right)^{\frac{6}{5}}\right] = n^{\frac{1}{2}} + 2^{-\frac{6}{5}} n^{\frac{1}{5}}$$

The rich maximizes

$$\max v_R(f,l) - \frac{1}{n} c(f,l)$$

so the solution from FOC would be

$$f_R = \left(\frac{n}{2}\right)^{\frac{3}{5}} \quad , \quad l_R = n^{\frac{3}{4}}$$

Since $n_P > n_R$, with the majority voting, the winning one is

$$f = n^{\frac{3}{4}} \quad , \quad l = \left(\frac{n}{2}\right)^{\frac{3}{5}} \quad , \quad t = n^{\frac{1}{2}} + 2^{-\frac{6}{5}} n^{\frac{1}{5}}$$

This is not Pareto optimal. Pareto optimal solution would be

$$\begin{aligned} \left(\frac{n}{2}\right)^{\frac{3}{5}} < f^{PO} < n^{\frac{3}{4}} \\ \left(\frac{n}{2}\right)^{\frac{3}{5}} < l^{PO} < n^{\frac{3}{4}} \end{aligned}$$