2006 Spring 1. Equilibrium under Uncertainty

(a) A has $\omega_A = (10, 10, 10, 6)$ and B has $\omega_B = (14, 6, 22, 10)$. Since the utility function $U^h$ is homothetic and identical, we can consider a representative agent with aggregate endowment $\omega = (24, 16, 32, 16)$. At the equilibrium, the price $p$ will be

$$p = \lambda \frac{\partial U}{\partial c} \bigg|_\omega = \lambda \left( \frac{1}{c_1}, \frac{\delta}{3c_{21}}, \frac{\delta}{3c_{22}}, \frac{\delta}{3c_{23}} \right) \bigg|_\omega = \lambda \left( \frac{1}{24}, \frac{\delta}{48}, \frac{\delta}{96}, \frac{\delta}{48} \right)$$

Since $p_1 = 1$, $\lambda = 24$, thus

$$p = \left( 1, \frac{\delta}{2}, \frac{\delta}{4}, \frac{\delta}{2} \right)$$

(b)

$$p = \left( 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{2} \right)$$

Thus, prices of two assets are

$$p_A = 1 \cdot 10 + \frac{1}{2} \cdot 10 + \frac{1}{4} \cdot 10 + \frac{1}{2} \cdot 6 = \frac{41}{2}$$

$$p_B = 1 \cdot 14 + \frac{1}{2} \cdot 6 + \frac{1}{4} \cdot 22 + \frac{1}{2} \cdot 10 = \frac{55}{2}$$

(c) Since the utility function is homothetic, every agent will consume a fraction of the aggregate endowment. The aggregate endowment can be replicated by a linear combination of two assets. So any Walrasian equilibrium allocation is achievable.

(Question: what is the allocation in (a)?)

(d) If we store 1 unit of commodity, trade in futures market yields a profit:

$$\text{profit} = p_{21} + p_{22} + p_{23} - p_1 = \frac{5}{4} \delta - 1$$

If this is less than or equal to 0, we will not store. This condition is

$$\delta \leq \frac{4}{5}$$
2006 Spring 2. Input and Output Prices

(a)\[ \max_{z_1, z_2} -r_1 z_1 - r_2 z_2 \quad \text{subject to } q = z_1^\alpha z_2^\beta \]

\[ \mathcal{L} = -r_1 z_1 - r_2 z_2 + \lambda z_1^\alpha z_2^\beta - q \]

The first order conditions are

\[ \frac{\partial \mathcal{L}}{\partial z_1} = -r_1 + \lambda \alpha z_1^{\alpha-1} z_2^\beta \leq 0 \quad (1) \]
\[ \frac{\partial \mathcal{L}}{\partial z_2} = -r_2 + \lambda \beta z_1^\alpha z_2^{\beta-1} \leq 0 \quad (2) \]
\[ \frac{\partial \mathcal{L}}{\partial z_2} = z_1^\alpha z_2^\beta - q \geq 0 \quad (3) \]

Assuming \( z_1, z_2, \lambda > 0 \), from (1) and (2), we have

\[ \frac{r_1}{r_2} = \frac{\alpha z_2}{\beta z_1} \]

Substituting into (3),

\[ z_1^\alpha \left( \frac{r_1}{r_2} \cdot \frac{\beta z_1}{\alpha} \right)^\beta = q \]

\[ z_1 = q^{\frac{\alpha}{\alpha + \beta}} \left( \frac{r_2}{r_1} \cdot \frac{\alpha}{\beta} \right)^{\frac{\beta}{\alpha + \beta}} \]
\[ z_2 = q^{\frac{\beta}{\alpha + \beta}} \left( \frac{r_1}{r_2} \cdot \frac{\beta}{\alpha} \right)^{\frac{\alpha}{\alpha + \beta}} \]

Therefore

\[ C(q, r) = r_1 z_1 + r_2 z_2 \]
\[ = q^{\frac{1}{\alpha + \beta}} r_1^{\frac{\alpha}{\alpha + \beta}} r_2^{\frac{\beta}{\alpha + \beta}} \left[ \left( \frac{\alpha}{\beta} \right)^{\frac{\beta}{\alpha + \beta}} + \left( \frac{\beta}{\alpha} \right)^{\frac{\alpha}{\alpha + \beta}} \right] \]
\[ = q^{\frac{1}{\alpha + \beta}} r_1^{\frac{\alpha}{\alpha + \beta}} r_2^{\frac{\beta}{\alpha + \beta}} \left( \frac{\alpha + \beta}{\beta^{\frac{1}{\alpha + \beta}} \alpha^{\frac{1}{\alpha + \beta}}} \right) \]
\[ = q^{\frac{1}{\alpha + \beta}} \left( \frac{\alpha}{\alpha + \beta} \right)^{\frac{\alpha}{\alpha + \beta}} \left( \frac{\beta}{\alpha + \beta} \right)^{\frac{\beta}{\alpha + \beta}} \]

(b) Hessian matrices for two production functions are

\[ H_A = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{8} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{8} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ -\frac{1}{16} & -\frac{1}{16} & -\frac{1}{16} & -\frac{1}{16} & -\frac{1}{16} \end{pmatrix} \]

\[^{1}\text{It can be easily shown that there is no solution where any of these is 0. If } \lambda = 0, \text{ (1) and (2) don’t hold for equality, so } z_1 = z_2 = 0. \text{ If either } z_1 = 0 \text{ or } z_2 = 0, \text{ (3) cannot be satisfied.}\]
\[ H_B = \begin{pmatrix} -\frac{1}{4}z_{B1}^{-\frac{3}{2}}z_{B2}^{\frac{1}{2}} & \frac{1}{2}z_{B1}^{-\frac{1}{2}}z_{B2}^{\frac{1}{2}} \\ \frac{1}{2}z_{B1}^{-\frac{1}{2}}z_{B2}^{\frac{1}{2}} & -\frac{1}{4}z_{B1}^{-\frac{3}{2}}z_{B2}^{\frac{1}{2}} \end{pmatrix} \]

Since \( \det(H_A) > 0 \) and \( \det(H_B) = 0 \), \( H_A \) is negative definite and \( H_B \) is negative semidefinite. Therefore, \( q_A \) is strictly concave and \( q_B \) is concave.

(c)
Define \( q_A^0 = f_A(z_A^0) \), \( q_A^1 = f_A(z_A^1) \), \( q_B^0 = f_B(z_B^0) \) and \( q_B^1 = f_B(z_B^1) \). Since \( (q_A^0, q_B^0) \) and \( (q_A^1, q_B^1) \) are production efficient, corresponding input vectors \( z_A^0, z_A^1, z_B^0 \), and \( z_B^1 \) satisfy
\[
\begin{align*}
z_{A1}^0 + z_{B1}^0 &= z_{A1}^1 + z_{B1}^1 = z_{A2}^0 + z_{B2}^0 = z_{A2}^1 + z_{B2}^1 = 1
\end{align*}
\]
Thus production of commodity A with input vector \((1 - \lambda)\) is feasible since commodity B with \((1 - \lambda)\) is also feasible since
\[
\begin{align*}
(1 - \lambda) (z_{A1}^0 + z_{B1}^0) + \lambda (z_{A1}^1 + z_{B1}^1) &= 1 \\
(1 - \lambda) (z_{A2}^0 + z_{B2}^0) + \lambda (z_{A2}^1 + z_{B2}^1) &= 1
\end{align*}
\]
Since \( q_A \) is strictly concave and \( q_B \) is concave,
\[
\begin{align*}
f_A ([1 - \lambda]z_A^0 + \lambda z_A^1) &> (1 - \lambda)f_A(z_A^0) + \lambda f_A(z_A^1) = (1 - \lambda)q_A^0 + \lambda q_A^1 \\
f_B ([1 - \lambda]z_B^0 + \lambda z_B^1) &\geq (1 - \lambda)f_B(z_B^0) + \lambda f_B(z_B^1) = (1 - \lambda)q_B^0 + \lambda q_B^1
\end{align*}
\]
Therefore,
\[
\begin{align*}
(1 - \lambda) (q_A^0, q_B^0) + \lambda (q_A^1, q_B^1) &< (f_A ([1 - \lambda]z_A^0 + \lambda z_A^1), f_B ([1 - \lambda]z_B^0 + \lambda z_B^1))
\end{align*}
\]
Since the right hand side is feasible production, \((1 - \lambda) (q_A^0, q_B^0) + \lambda (q_A^1, q_B^1)\) is in the interior for all \( \lambda \in (0, 1) \).

(d)
Note that \( A \) is more input 1 intensive since
\[
\begin{align*}
MRTS_A|_{(1,1)} &= \frac{1}{\frac{1}{2}z_{A1}^{-\frac{3}{2}}z_{A2}^{\frac{1}{2}}} = 2 > 1 = \frac{1}{\frac{1}{2}z_{B1}^{-\frac{3}{2}}z_{B2}^{\frac{1}{2}}} = MRTS_B|_{(1,1)}
\end{align*}
\]
Thus, along the production efficient allocation curve, \( MRTS_A = MRTS_B \) increases. If all endowments are used to produce commodity A, the input price ratio will be equal to \( MRTS_A|_{(1,1)} = 2 \), and if all endowments are used to produce commodity B, the input price ratio will be equal to \( MRTS_B|_{(1,1)} = 1 \). Therefore, the range of input price ratio is
\[
1 \leq \frac{r_1}{r_2} \leq 2
\]

(e)
Note that there is no 1-1 relationship between \( \frac{p_A}{p_B} \) and \( \frac{r_1}{r_2} \) since \( q_A \) is not constant returns to scale. Using the formula obtained in (a),
\[
\frac{p_A}{p_B} = \frac{MC_A}{MC_B} = (2q_A)^\frac{1}{2} \left( \frac{r_1}{r_2} \right)^\frac{1}{2}
\]
If the left hand side is always greater than the right hand side (for all feasible \( q_A \)), then the economy will specialize in the production of commodity A. At such state, \( q_A = 1 \), and \( \frac{r_1}{r_2} = 2 \), thus the condition under which only commodity A is produced is²
\[
\frac{p_A}{p_B} \geq \sqrt{2}
\]

²We have to assume that there is only one firm producing each commodity. Note that if either \( r_1 \) or \( r_2 \) is given, \( p_A \) also can be calculated from \( p_A = MC_A \). Note also that this economy cannot specialize in the production of commodity B. Think why.