2006 Spring 1. Equilibrium under Uncertainty

(a)
A has $\omega_A = (10, 10, 10, 6)$ and B has $\omega_B = (14, 6, 22, 10)$. Since the utility function U^h is homothetic and identical, we can consider a representative agent with aggregate endowment $\omega = (24, 16, 32, 16)$. At the equilibrium, the price p will be

$$p = \lambda \left.\frac{\partial U}{\partial c}\right|_{\omega} = \lambda \left.\left(\frac{1}{c_1}, \frac{\delta}{3c_{21}}, \frac{\delta}{3c_{22}}, \frac{\delta}{3c_{23}}\right)\right|_{\omega} = \lambda \left(\frac{1}{24}, \frac{\delta}{48}, \frac{\delta}{96}, \frac{\delta}{48}\right)$$

Since $p_1 = 1$, $\lambda = 24$, thus

$$p = \left(1, \frac{\delta}{2}, \frac{\delta}{4}, \frac{\delta}{2}\right)$$

(b)

$$p = \left(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{2}\right)$$

Thus, prices of two assets are

$$p_A = 1 \cdot 10 + \frac{1}{2} \cdot 10 + \frac{1}{4} \cdot 10 + \frac{1}{2} \cdot 6 = \frac{41}{2}$$
$$p_B = 1 \cdot 14 + \frac{1}{2} \cdot 6 + \frac{1}{4} \cdot 22 + \frac{1}{2} \cdot 10 = \frac{55}{2}$$

(c)
Since the utility function is homothetic, every agent will consume a fraction of the aggregate endowment. The aggregate endowment can be replicated by a linear combination of two assets. So any Walrasian equilibrium allocation is achievable.
(Question: what is the allocation in (a)?)

(d)
If we store 1 unit of commodity, trade in futures market yields a profit:

$$\text{profit} = p_{21} + p_{22} + p_{23} - p_1 = \frac{5}{4}\delta - 1$$

If this is less than or equal to 0, we will not store. This condition is

$$\delta \leq \frac{4}{5}$$

2006 Spring 2. Input and Output Prices

(a)

$$\max_{z_1,z_2} -r_1z_1 - r_2z_2 \quad \text{subject to } q = z_1^\alpha z_2^\beta$$

$$\mathcal{L} = -r_1z_1 - r_2z_2 + \lambda z_1^\alpha z_2^\beta - q$$

The first order conditions are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial z_1} &= -r_1 + \lambda\alpha z_1^{\alpha-1} z_2^\beta \le 0 && (1) \\ \frac{\partial \mathcal{L}}{\partial z_2} &= -r_2 + \lambda\beta z_1^{\alpha} z_2^{\beta-1} \le 0 && (2) \\ \frac{\partial \mathcal{L}}{\partial z_2} &= z_1^\alpha z_2^\beta - q \ge 0 && (3) \end{aligned}$$

Assuming $z_1, z_2, \lambda > 0$[1], from (1) and (2), we have

$$\frac{r_1}{r_2} = \frac{\alpha z_2}{\beta z_1}$$

Substituting into (3),

$$z_1^\alpha \left(\frac{r_1}{r_2} \cdot \frac{\beta z_1}{\alpha}\right)^\beta = q$$

$$\begin{aligned} z_1 &= q^{\frac{1}{\alpha+\beta}} \left(\frac{r_2}{r_1} \cdot \frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} \\ z_2 &= q^{\frac{1}{\alpha+\beta}} \left(\frac{r_1}{r_2} \cdot \frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} \end{aligned}$$

Therefore

$$\begin{aligned} C(q,r) &= r_1z_1 + r_2z_2 \\ &= q^{\frac{1}{\alpha+\beta}} r_1^{\frac{\alpha}{\alpha+\beta}} r_2^{\frac{\beta}{\alpha+\beta}} \left[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right] \\ &= q^{\frac{1}{\alpha+\beta}} r_1^{\frac{\alpha}{\alpha+\beta}} r_2^{\frac{\beta}{\alpha+\beta}} \left(\frac{\alpha+\beta}{\beta^{\frac{\beta}{\alpha+\beta}} \alpha^{\frac{\alpha}{\alpha+\beta}}}\right) \\ &= q^{\frac{1}{\alpha+\beta}} \frac{r_1^{\frac{\alpha}{\alpha+\beta}} r_2^{\frac{\beta}{\alpha+\beta}}}{\left(\frac{\alpha}{\alpha+\beta}\right)^{\frac{\alpha}{\alpha+\beta}} \left(\frac{\beta}{\alpha+\beta}\right)^{\frac{\beta}{\alpha+\beta}}} \end{aligned}$$

(b)
Hessian matrices for two production fuctions are

$$H_A = \begin{pmatrix} -\frac{1}{4} z_{A1}^{-\frac{3}{2}} z_{A2}^{\frac{1}{4}} & \frac{1}{8} z_{A1}^{-\frac{1}{2}} z_{A2}^{-\frac{3}{4}} \\ \frac{1}{8} z_{A1}^{-\frac{1}{2}} z_{A2}^{-\frac{3}{4}} & -\frac{3}{16} z_{A1}^{\frac{1}{2}} z_{A2}^{-\frac{7}{4}} \end{pmatrix}$$

[1] It can be easily shown that there is no solution where any of these is 0. If $\lambda = 0$, (1) and (2) don't hold for equality, so $z_1 = z_2 = 0$. If either $z_1 = 0$ or $z_2 = 0$, (3) cannot be satisfied.

$$H_B = \begin{pmatrix} -\frac{1}{4} z_{B1}^{-\frac{3}{2}} z_{B2}^{\frac{1}{2}} & \frac{1}{4} z_{B1}^{-\frac{1}{2}} z_{B2}^{-\frac{1}{2}} \\ \\ \frac{1}{4} z_{B1}^{-\frac{1}{2}} z_{B2}^{-\frac{1}{2}} & -\frac{1}{4} z_{B1}^{\frac{1}{2}} z_{B2}^{-\frac{3}{2}} \end{pmatrix}$$

Since $\det(H_A) > 0$ and $\det(H_B) = 0$, H_A is negative definite and H_B is negative semidefinite. Therefore, q_A is strictly concave and q_B is concave.

(c)

Define $q_A^0 = f_A\left(z_A^0\right)$, $q_A^1 = f_A\left(z_A^1\right)$, $q_B^0 = f_B\left(z_B^0\right)$ and $q_B^1 = f_B\left(z_B^1\right)$. Since $\left(q_A^0, q_B^0\right)$ and $\left(q_A^1, q_B^1\right)$ are production efficient, corresponding input vectors z_A^0, z_A^1, z_B^0, and z_B^1 satisfy

$$z_{A1}^0 + z_{B1}^0 = z_{A1}^1 + z_{B1}^1 = z_{A2}^0 + z_{B2}^0 = z_{A2}^1 + z_{B2}^1 = 1$$

Thus production of commodity A with input vector $(1-\lambda)z_A^0 + \lambda z_A^1$ and of commodity B with $(1-\lambda)z_B^0 + \lambda z_B^1$ is also feasible since

$$(1-\lambda)\left(z_{A1}^0 + z_{B1}^0\right) + \lambda\left(z_{A1}^1 + z_{B1}^1\right) = 1$$
$$(1-\lambda)\left(z_{A2}^0 + z_{B2}^0\right) + \lambda\left(z_{A2}^1 + z_{B2}^1\right) = 1$$

Since q_A is strictly concave and q_B is concave,

$$f_A\left([1-\lambda]z_A^0 + \lambda z_A^1\right) > (1-\lambda) f_A\left(z_A^0\right) + \lambda f_A\left(z_A^1\right) = (1-\lambda)q_A^0 + \lambda q_A^1$$
$$f_B\left([1-\lambda]z_B^0 + \lambda z_B^1\right) \geq (1-\lambda) f_B\left(z_B^0\right) + \lambda f_A\left(z_B^1\right) = (1-\lambda)q_B^0 + \lambda q_B^1$$

Therefore,

$$(1-\lambda)\left(q_A^0, q_B^0\right) + \lambda\left(q_A^1, q_B^1\right) < \left(f_A\left[(1-\lambda)z_A^0 + \lambda z_A^1\right], f_B\left[(1-\lambda)z_B^0 + \lambda z_B^1\right]\right)$$

Since the right hand side is feasible production, $(1-\lambda)\left(q_A^0, q_B^0\right) + \lambda\left(q_A^1, q_B^1\right)$ is in the interior for all $\lambda \in (0,1)$.

(d)

Note that A is more input 1 intensive since

$$MRTS_A|_{(1,1)} = \left.\frac{\frac{1}{2} z_{A1}^{-\frac{1}{2}} z_{A2}^{\frac{1}{4}}}{\frac{1}{4} z_{A1}^{\frac{1}{2}} z_{A2}^{-\frac{3}{4}}}\right|_{(1,1)} = 2 > 1 = \left.\frac{\frac{1}{2} z_{B1}^{-\frac{1}{2}} z_{B2}^{\frac{1}{2}}}{\frac{1}{2} z_{B1}^{\frac{1}{2}} z_{B2}^{-\frac{1}{2}}}\right|_{(1,1)} = MRTS_B|_{(1,1)}$$

Thus, along the production efficient allocation curve, $MRTS_A = MRTS_B$ increases. If all endowments are used to produce commodity A, the input price raio will be equal to $MRTS_A|_{(1,1)} = 2$, and if all endowments are used to produce commoidty B, the input price ratio will be equal to $MRTS_B|_{(1,.1)} = 1$. Therefore, the range of input price ratio is

$$1 \leq \frac{r_1}{r_2} \leq 2$$

(e)

Note that there is no 1-1 relationship between $\frac{p_A}{p_B}$ and $\frac{r_1}{r_2}$ since q_A is not constant returns to scale. Using the formula obtained in (a),

$$\frac{p_A}{p_B} = \frac{MC_A}{MC_B} = (2q_A)^{\frac{1}{3}} \left(\frac{r_1}{r_2}\right)^{\frac{1}{6}}$$

If the left hand side is always greater than the right hand side (for all feasible q_A), then the economy will specialize in the production of commodity A. At such state, $q_A = 1$, and $\frac{r_1}{r_2} = 2$, thus the condition under which only commodity A is produced is[2]

$$\frac{p_A}{p_B} \geq \sqrt{2}$$

[2] We have to assume that there is only one firm producing each commodity. Note that if either r_1 or r_2 is given, p_A also can be calculated from $p_A = MC_A$. Note also that this economy cannot specialize in the production of commodity B. Think why.