41. Left unsolved

42. (a) Note that \( P(|S_{n+k} - S_n| \geq k \) for infinitely many \( n \) \) \( = 1 \iff P(|S_{n+k} - S_n| \geq k \) i.o. \( ) = 1 \iff P(|S_{n+k} - S_n| < k \) fin.) \( = 0 \). Consider the sequence \( |S_{2k} - S_k|, |S_{3k} - S_{2k}|, \ldots \). These are iid with \( P(|S_{tk} - S_{(t-1)k}| = k) = \frac{1}{2k-1} \) for any \( t \). This implies that \( P(|S_{tk} - S_{(t-1)k}| < k \) fin.) \( = 0 \). Therefore \( P(|S_{n+k} - S_n| < k \) fin.) \( = 0 \). This holds for any \( k \).

(b) Consider \( \omega \in \{ |S_{n+2m+1} - S_n| \geq 2m + 1 \) i.o. \}. For this \( \omega \), there is \( N \) such that \( |S_{N+2m+1} - S_N| = 2m + 1 \). Since \( |S_{N+2m+1}| + |S_N| \geq |S_{N+2m+1} - S_N| \), we have either \( |S_{N+2m+1}| > m \) or \( |S_N| > m \). Therefore, for this \( \omega \), it never happens that \( |S_n| \leq m \) for all \( n \). By (a), \( P(|S_n| \leq m \) for all \( n \) \) \( = 1 - P(|S_{n+2m+1} - S_n| \geq 2m + 1 \) i.o. \) \( = 0 \). This holds for every \( m \).

(c) By the same logic with (a) and (b), we can prove that \( P(S_n \leq m \) for all \( n \) \) \( = 0 \) for every \( m \). In other words, \( P(S_n > m \) for some \( n \) \) \( = 1 \) for every \( m \), so let \( m \to \infty \), then \( P(\sup S_n = \infty) = 1 \). Similarly, we have \( P(\inf S_n = -\infty) = 1 \). It follows that \( P(\sup S_n = \infty, \inf S_n = -\infty) = 1 \). (When \( P(A) = P(B) = 1 \), \( P(A \cap B) = P(A) + P(B) - P(A \cup B) = 1 \). Do we need to mention it?)

43. Let \( \tau \) denote the first time such that \( |S_j - S_m| > 2\varepsilon \), and \( A_j := \{ \tau = j \} \), then \( \bigcup_{j=m+1}^n A_j = \{ \max_{m<j\leq n} |S_j - S_m| > 2\varepsilon \} \). Note that \( A_j \in \sigma(X_{m+1}, \ldots, X_j) \) and \( |S_n - S_j| \in \sigma(X_{j+1}, \ldots, X_n) \), and thus that \( A_j \) and \( |S_n - S_j| \) are independent. So \( P(A_j \cap \{|S_n - S_j| \leq \varepsilon\}) = P(A_j)P(|S_n - S_j| \leq \varepsilon) \). Note that \( A_j \cap \{|S_n - S_j| \leq \varepsilon\} \subset \{|S_n - S_m| > \varepsilon\} \), and thus that \( A_j \cap \{|S_n - S_j| \leq \varepsilon\} \subset A_j \cap \{|S_n - S_m| > \varepsilon\} \). So \( P(A_j)P(|S_n - S_j| \leq \varepsilon) \leq P(A_j \cap \{|S_n - S_m| > \varepsilon\}) \). Sum this over \( j = m+1, \ldots, n \), then

\[
\sum_{j=m+1}^n P(A_j)P(|S_n - S_j| \leq \varepsilon) \leq \sum_{j=m+1}^n P(A_j \cap \{|S_n - S_m| > \varepsilon\})
\]

Disjointness of \( A_j \) implies that \( \text{RHS} = P(\bigcup_j A_j \cap \{|S_n - S_m| > \varepsilon\}) \leq P(\{|S_n - S_m| > \varepsilon\}) \), and also that \( \text{LHS} \geq \sum_{j=m+1}^n P(A_j)\min_{k \leq n} P(|S_n - S_k| \leq \varepsilon) \). So we have

\[
P\left(\max_{m<j\leq n} |S_j - S_m| > 2\varepsilon\right) \min_{m<k\leq n} P(|S_n - S_k| \leq \varepsilon) \leq P(|S_n - S_m| > \varepsilon)
\]

This holds for any \( \varepsilon > 0 \).

44. From the result of 43,

\[
P\left(\max_{j\leq n} |S_j| > n\varepsilon\right) \min_{k\leq n} P\left(|S_n - S_k| \leq \frac{n\varepsilon}{2}\right) \leq P\left(|S_n| > \frac{n\varepsilon}{2}\right)
\]

Note that \( \text{RHS} = P(\frac{|S_n|}{n} > \frac{\varepsilon}{2}) \to 0 \), so \( \text{LHS} \to 0 \). Since \( X_n \) are iid, \( \min_{k\leq n} P(|S_n - S_k| \leq \frac{n\varepsilon}{2}) = \min_{k\leq n} P(|S_k| \leq \frac{n\varepsilon}{2}) \). Since \( P(\frac{|S_n|}{n} \leq \frac{\varepsilon}{2}) \to 1 \) for any \( k \) and \( \varepsilon > 0 \), \( \min_{k\leq n} P(|S_n - S_k| \leq \frac{n\varepsilon}{2}) \to 1 \),
which implies that \( P(\max_{j \leq n} |S_j| > n\varepsilon) \to 0 \). Note that \( P(\max_{j \leq n} |S_j| > n\varepsilon) \leq P(\max_{j \leq n} |S_j| > n\varepsilon). So we have

\[
P \left( \frac{\max_{m \leq n} S_m}{n} > \varepsilon \right) \to 0
\]

This holds for any \( \varepsilon > 0 \), so \( \frac{\max_{m \leq n} S_m}{n} \to 0 \) in probability.

**45.** (a) \( \{\max_{m^\alpha \leq n \leq (m+1)^\alpha} \frac{|S_n|}{n^p} \geq \varepsilon\} \subset \{\max_{m^\alpha \leq n \leq (m+1)^\alpha} \frac{|S_n|}{m^{2\alpha p^2}} \geq \varepsilon\} \subset \{\max_{n \leq (m+1)^\alpha} \frac{|S_n|}{m^{2\alpha p^2}} \geq \varepsilon\}. \) By Kolmogorov’s inequality,

\[
P \left( \max_{m^\alpha \leq n \leq (m+1)^\alpha} \frac{|S_n|}{n^p} \geq \varepsilon \right) \leq P \left( \max_{n \leq (m+1)^\alpha} |S_n| \geq m^{2\alpha p^2} \varepsilon \right) \leq \frac{\mathcal{V}(S(m+1)^\alpha)}{m^{2\alpha p^2}} = \left( 1 + \frac{1}{m} \right)^{\alpha} \mathcal{V}(X_1)
\]

As \( n \to \infty, m \to \infty \) for fixed \( \alpha \), so RHS \( \to 0 \), which implies that \( \frac{S_n}{n^p} \to 0 \) a.s.

(b) Note that \( \sum E(X_k^2) = \sum \frac{E(X_k^2)}{k^p} = E(X_k^2) \sum \frac{1}{k^p} < \infty \) when \( p > \frac{1}{2} \). By L² R.S. theorem, \( \sum \frac{X_k}{k^p} \) converges a.s. Then it follows by Kronecker’s lemma that \( \frac{1}{n^p} \sum_{k=1}^n X_k \) converges a.s.

**46.** (a) implies (b): (a) implies \( P(X_n > 1 \text{ i.o.}) = 0 \), which further implies \( \sum P(X_n > 1) < \infty \). Note also that (a) implies \( \sum E(X_n) < \infty \), take \( \bar{X}_n = X_n \cdot 1_{X_n \leq 1} \), then (b) follows by summing up the two inequalities.

(b) implies (c): Note that \( P(X_n > 1) = E(1_{X_n > 1}) \geq E(\frac{X_n}{1+X_n} \cdot 1_{X_n > 1}) \), and also that \( E(X_n \cdot 1_{X_n \leq 1}) \geq E(\frac{X_n}{1+X_n} \cdot 1_{X_n \leq 1}) \). (c) follows by adding the two inequalities and summing over \( n \).

(c) implies (a): \( \infty > E(\frac{X_n}{1+X_n}) \geq E(\frac{X_n}{1+X_n} \cdot 1_{X_n \leq 1}) \geq E(\frac{X_n}{2} \cdot 1_{X_n \leq 1}) = \frac{1}{2} E(\bar{X}_n) \) implies \( \sum X_n < \infty \) a.s.

**47.** (a) Define \( \bar{X}_n := X_n \cdot 1_{|X_n| \leq 1} \). Since \( X_n \) is symmetric, \( \sum X_n \) converges a.s. if and only if \( \sum V(\bar{X}_n) < \infty \). Note that \( V(\bar{X}_n) = \min\{\frac{1}{2}, \frac{1}{3n^p}\} \), and thus that \( \sum V(\bar{X}_n) \) converges only when \( p > \frac{1}{2} \). When \( p = \frac{1}{2} \), \( \sum X_n \) is distributed on the interval \( [\sum -\frac{1}{\sqrt{n}}, \sum \frac{1}{\sqrt{n}}] \), which expands at the rate of \( \sqrt{n} \). As \( n \to \infty \), \( \sup |\sum X_n| \to \infty \) but at the rate of \( \log n \). (Should we prove it?)

(b) Use the same \( \bar{X}_n \). Since \( X_n \geq 0 \), \( \sum X_n \) converges a.s. if and only if \( \sum E(\bar{X}_n) < \infty \). Note that \( E(\bar{X}_n) = \min\{\frac{1}{2}, \frac{1}{2n^p}\} \), and thus that \( \sum E(\bar{X}_n) \) converges only when \( p > 1 \). When \( p = 1 \), \( \sum X_n \) increases at the rate of \( \log n \).

(c) In (a), \( X_{n+1} \) might have a different sign with \( \sum_{k=1}^n X_k \), so \( \sum X_n \) does not expand so fast, while in (b), \( \sum X_n \) is an increasing sequence, so expands faster than in (a).

48. I will skip.