Final Exam
Dec. 14th, 2007

Answer all questions, showing your work. Partially correct work will be given partial credit. All questions have the same value, but not necessarily the same level of difficulty. Good luck!

Question 1: A group of young and old people was divided into three committees. Committee A has 2 old and 3 young. Committee B has 3 old and 1 young, and Committee C has 1 old and 3 young. An individual was taken at random from Committee A and put into Committee B. Then an individual from Committee B was taken at random and put into Committee C. Finally, an individual was then taken at random from Committee C. If the individual taken from Committee C was young, what is the probability that the one taken from Committee A and into Committee B was young?

Question 2: Suppose that \((X_1, X_2)\) is normally distributed with mean \((0, 0)\) and variance-covariance matrix

\[
I = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

Let

\[
Y = \begin{cases}
X_1^2 & \text{if } X_1 > 0 \text{ and } X_2 > 0 \\
X_2^2 & \text{otherwise}
\end{cases}
\]

Derive the probability density of \(Y\).

Question 3: Suppose that \(\{(X_n, Z_n)\}_{n=1}^{\infty}\) is a sequence such that for each \(n\), the distribution of \(X_n\) given \(Z_n = z_n\) is \(\exp(z_n)\) and the distribution of \(Z_n\) is uniform between \(a_n\) and \(b_n\), where \(b_n > a_n > 0\). That is, for \(x > 0\),

\[
f_{X_n|Z_n=z}(x) = ze^{-zx}
\]

and for \(x \leq 0\), \(f_{X_n|Z_n=z}(x) = 0\) and for each \(n\),

\[
f_{Z_n}(t) = \begin{cases}
\frac{1}{b_n-a_n} & a_n < t < b_n \\
0 & \text{otherwise}
\end{cases}
\]

Suppose that, as \(n \to \infty\), \(a_n \to a\) and \(b_n \to b\), where \(b > a > 0\). What is the limiting distribution of the sequence \(\{X_n\}\)?
Question 4: The two dimensional random vector \( X \) has the following distribution:

\[
X = \begin{cases} 
\begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{with probability } p_1 \\
\begin{pmatrix} 0 \\ 1 \end{pmatrix} & \text{with probability } p_2 \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{with probability } 1 - p_1 - p_2 
\end{cases}
\]

Of 100 independent observations, 30 fell into the first case, 40 into the second, and 30 into the third. What is the Maximum Likelihood Estimator for \((p_1, p_2)\)? Derive the asymptotic distribution of this estimator. Provide approximate 95% confidence intervals for \( \theta = p_1 - p_2 \) and for \( \lambda = p_1^2 \).

Question 5: Suppose that the value of a random variable \( X \) is known to be distributed as \( N(\mu_1, \sigma^2) \) for men and as \( N(\mu_2, \sigma^2) \) for women. A sample \( X_{ij} \) \((i = 1, \ldots, n; j = 1, 2)\) of independent observations was taken for \( n \) men and \( n \) women. Derive the equations for the Maximum Likelihood Estimators, \( \hat{\mu}_1, \hat{\mu}_2, \) and \( \hat{\sigma}^2 \) of \( \mu_1, \mu_2 \) and of \( \sigma^2 \). Answer the following directly from these equations: Is \( \hat{\mu}_j \) an unbiased estimator for \( \mu_j \)? Is \( \hat{\mu}_j \) a consistent estimator for \( \mu_j \)? Is \( \hat{\sigma}^2 \) a consistent estimator for \( \sigma^2 \)? Is \( \hat{\sigma}^2 \) an unbiased estimator for \( \sigma^2 \)?

(Bonus) Let \( j = 1, \ldots, m \) instead. Fix \( n = 2 \) and let \( m \) grow to \( \infty \). How would your answer to those questions change?

Question 6: Let \( X_1, \ldots, X_n \) and \( Y_1, \ldots, Y_m \) be independently distributed with

\[
f_{X_i}(x) = \begin{cases} 
\theta x^{\theta-1} & 0 < x < 1 \\
0 & \text{otherwise} 
\end{cases}
\]

and

\[
f_{Y_j}(y) = \begin{cases} 
\lambda y^{\lambda-1} & 0 < y < 1 \\
0 & \text{otherwise} 
\end{cases}
\]

Describe how to test the null hypothesis, \( H_0 : \theta = \lambda \) versus the alternative \( H_1 : \theta \neq \lambda \) using a Likelihood Ratio test and a Wald test. How would you test this using a Lagrange Multiplier test? How would you test \( H_0 : \theta = 1 \) versus \( H_1 : \theta \neq 1 \) using a Lagrange Multiplier test?