Question 1:

1.a. Show that if $\Gamma$ and $\Theta$ are $\sigma-$algebras on $S$, the intersection of $\Gamma$ and $\Theta$ is also a $\sigma-$algebra on $S$.

1.b. Let $A$ denote a collection of subsets in $S$. Let $\Phi$ denote the intersection of all $\sigma-$algebras that contain $A$. Show that $\Phi$ is not empty and that it is a $\sigma-$algebra. What is the smallest $\sigma-$algebra that contains $A$?

1.c. Let $S$ denote the real line, and suppose that $A$ is the set of all open sets in $S$. Let $\Phi$ denote the smallest $\sigma-$algebra that contains $A$. Does the set $\{0\}$ belong to $\Phi$? Does the interval $(a, b]$ (where $a < b$) belong to $\Phi$? Explain.

Question 2:

Let $C_1, ..., C_n$ be sets in a sample space, $S$.

2.a. Prove the Inclusion-Exclusion Formula:

$$P\left(\bigcup_{i=1}^{k} C_i\right) = p_1 - p_2 + p_3 - \ldots + (-1)^{k+1} p_k$$

where $p_i$ is the sum of all possible intersections involving $i$ sets.

2.b. Prove Boole’s inequality:

$$P\left(\bigcup_{i=1}^{n} C_i\right) \leq \sum_{i=1}^{n} P(C_i)$$

Question 3:

Let $(S_X, \Gamma_X, P_X)$ denote the probability space induced by a random variable, $X$, on $(S, \Gamma, P)$. Let $A$ and $B$ be elements in $\Gamma_X$ and $A_1, A_2, \ldots$ be a sequence in $\Gamma_X$. Determine whether the following are true or false. Prove your answers.
3.a. $X^{-1}(A \cup B) = X^{-1}(A) \cup X^{-1}(B)$

3.b. $X^{-1}(\bigcup_{i=1}^{\infty} A_i) = \bigcup_{i=1}^{\infty} X^{-1}(A_i)$

3.c. $X^{-1}(\bigcap_{i=1}^{\infty} A_i) = \bigcap_{i=1}^{\infty} X^{-1}(A_i)$

3.d. $X^{-1}(A^c) = [X^{-1}(A)]^c$

Question 4:

Solve questions 1.2.17, 1.4.2 ad 1.4.27 from Hogg, McKean, and Craig.