Q1. (a)

We are required to show that $\Gamma \cap \Theta$ satisfies

1. $S \in \Gamma \cap \Theta$

2. $A \in \Gamma \cap \Theta$ implies $A^c \in \Gamma \cap \Theta$, and

3. $A_1, A_2, \cdots \in \Gamma \cap \Theta$ implies $\bigcup_{i=1}^{\infty} A_i \in \Gamma \cap \Theta$.

For (1), since $\Gamma$ and $\Theta$ are $\sigma$-algebras on $S$, $S \in \Gamma$ and $S \in \Theta$, and thus by definition, $S \in \Gamma \cap \Theta$.

For (2), pick any $A \in \Gamma \cap \Theta$, then $A \in \Gamma$ and $A \in \Theta$. This implies $A^c \in \Gamma$ and $A^c \in \Theta$, and thus $A^c \in \Gamma \cap \Theta$.

For (3), pick any sequence of sets $A_1, A_2, \cdots \in \Gamma \cap \Theta$. Obviously $A_i \in \Gamma$ for all $i = 1, 2, \cdots$. By definition of $\sigma$-algebra, $\bigcup_{i=1}^{\infty} A_i \in \Gamma$. The same logic can be applied to $\Theta$, so $\bigcup_{i=1}^{\infty} A_i \in \Gamma \cap \Theta$. 

(b)

Since any $\sigma$-algebra contains $S$, $\Phi$ contains $S$, and thus is not empty. By the result of (a), any intersection of two $\sigma$-algebras on $S$ is a $\sigma$-algebra on $S$. Therefore, $\Phi$ is a $\sigma$-algebra on $S$. Finally, $\Phi$ is the smallest $\sigma$-algebra that contains $A$. \footnote{This is straightforward, but can be proved. Suppose not. Then there exists some $\sigma$-algebra $\Theta$ that is smaller than $\Phi$. By definition of $\Phi$, however, $\Phi \subset \Theta$, which is a contradiction.}

(c)

$\{0\}$ belongs to $\Phi$. Let $B = \{x : x \neq 0\}$. Obviously $B \in A$, and thus $B \in \Phi$. By the definition of $\sigma$-algebra, its complement $\{0\}$ belongs to $\Phi$.

$(a, b]$ also belongs to $\Phi$. Since all open sets belong to $\Phi$, so do their complements. In other words, all closed sets belong to $\Phi$. Let $C_n = [a + \frac{1}{n}(b - a), b]$, then,

$$
\Phi \ni \bigcup_{i=1}^{\infty} C_n = \bigcup_{i=1}^{\infty} \left[a + \frac{1}{n}(b - a), b\right] = (a, b]
$$

Q2. (a)

Recall that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, which was proved in the class. Use mathematical induction. For $k = 1$, the equation is clearly true. For $k = 2$, the above formula directly applies.

Suppose the equation holds for $k = 1, \cdots, n$.

$$
P\left(\bigcup_{i=1}^{n+1} C_i\right) = P\left(\bigcup_{i=1}^{n} C_i \cup C_{n+1}\right)
$$

$$
= P\left(\bigcup_{i=1}^{n} C_i\right) + P(C_{n+1}) - P\left(\bigcup_{i=1}^{n} C_i \cap C_{n+1}\right)
$$
We have
\[ P \left( \bigcup_{i=1}^{n} C_i \right) = q_1 - q_2 + q_3 - \cdots + (-1)^{n+1} q_n \]
and
\[ P \left( \left[ \bigcup_{i=1}^{n} C_i \right] \cap C_{n+1} \right) = P \left( \bigcup_{i=1}^{n} [C_i \cap C_{n+1}] \right) = r_1 - r_2 + r_3 - \cdots + (-1)^{n+1} r_n \]
where \( q_m \) denotes the sum of probabilities of all possible intersections including \( m \) sets of \( C_i \), and \( r_m \) denotes the sum of probabilities of all possible intersections including \( m \) sets of \( C_i \cap C_{n+1} \). It is easy to check that
\[
q_1 + P (C_{n+1}) = p_1 \\
q_2 + r_1 = p_2 \\
\vdots \\
q_n + r_{n-1} = p_n \\
r_n = p_{n+1}
\]
Therefore,
\[ P \left( \bigcup_{i=1}^{n+1} C_i \right) = p_1 - p_2 + p_3 - \cdots + (-1)^{n+1} p_n + (-1)^{n+2} p_{n+1} \]
The equation holds for \( k = n + 1 \), so we are done.

(b) The inequality is clearly true when \( n = 1 \). For \( n = 2 \),
\[ P (C_1 \cup C_2) = P (C_1) + P (C_2) - P (C_1 \cap C_2) \leq P (C_1) + P (C_2) \]
since \( P (C_1 \cap C_2) \geq 0 \) by the definition of probability set function \( P \). Using this repeatedly,
\[
P \left( \bigcup_{i=1}^{n} C_i \right) \leq P (C_1) + P \left( \bigcup_{i=2}^{n} C_i \right) \\
\leq P (C_1) + P (C_2) + P \left( \bigcup_{i=3}^{n} C_i \right) \\
\vdots \\
\leq P (C_1) + P (C_2) + \cdots + P (C_n) \\
= \sum_{i=1}^{n} P (C_i)
\]
which completes the proof.
Q3. (a) True.

\[ c \in X^{-1}(A \cup B) \iff X(c) \in A \cup B \]
\[ \iff \text{either } X(c) \in A \text{ or } X(c) \in B \]
\[ \iff \text{either } c \in X^{-1}(A) \text{ or } c \in X^{-1}(B) \]
\[ \iff c \in X^{-1}(A) \cup X^{-1}(B) \]

(b) True.
It is easily verified by applying (a) repeatedly.

(c) True.

\[ c \in X^{-1}\left(\bigcap_{i=1}^{\infty} A_i\right) \iff X(c) \in \bigcap_{i=1}^{\infty} A_i \]
\[ \iff X(c) \in A_i \text{ for all } i = 1, 2, \cdots \]
\[ \iff c \in X^{-1}(A_i) \text{ for all } i = 1, 2, \cdots \]
\[ \iff c \in \bigcap_{i=1}^{\infty} X^{-1}(A_i) \]

(d) True.

\[ c \in X^{-1}(A^c) \iff X(c) \in A^c \]
\[ \iff X(c) \notin A \]
\[ \iff c \notin X^{-1}(A) \]
\[ \iff c \in [X^{-1}(A)]^c \]

HMC 1.2.17.
The report is false. Let \( A \) be the set of people who hurt a hip, \( B \) the set of those who hurt an arm, and \( C \) the set of those who hurt a hip. The report says \( n(A) = 8, n(B) = 6, n(C) = 5 \) and \( n(A \cap B) = 3, n(A \cap C) = 2, n(B \cap C) = 1, n(A \cap B \cap C) = 0 \). Then,

\[ 11 \geq n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C) = 13 \]

which is a contradiction.

HMC 1.4.2.
By definition,

\[ P(C_1 \cap C_2) = P(C_1) P(C_2 | C_1) \]
\[ P(C_1 \cap C_2 \cap C_3) = P(C_1 \cap C_2) P(C_3 | C_1 \cap C_2) \]
\[ P(C_1 \cap C_2 \cap C_3 \cap C_4) = P(C_1 \cap C_2 \cap C_3) P(C_4 | C_1 \cap C_2 \cap C_3) \]
Multiplying side by side and cancelling out yields the desired equation.

**HMC 1.4.27.**

Let $R$ be the event of the red tulip bulb being taken, and $Y$ that of the yellow tulip. Also $A$ be the event of the former type of box being selected, and $B$ that of the latter type of box.

\[
P(A) = .6 \quad \text{with} \quad P(R|A) = .2 \quad \text{and} \quad P(Y|A) = .8 \\
P(B) = .4 \quad \text{with} \quad P(R|B) = .6 \quad \text{and} \quad P(Y|B) = .4
\]

(a)

Note that $A$ and $B$ are a partition of the sample space $S$. So

\[
P(Y) = P(Y|A)P(A) + P(Y|B)P(B) = .48 + .16 = .64
\]

(b)

Applying Bayes Rule,

\[
P(A|Y) = \frac{P(A \cap Y)}{P(Y)} = \frac{P(Y|A)P(A)}{P(Y|A)P(A) + P(Y|B)P(B)} = \frac{.48}{.64} = .75
\]