1. Multivariate Normal Distribution

Let $Z_i \sim iid \ N(0, 1)$ (independent and identically distributed). Consider $Z = (Z_1, \cdots, Z_n)'$. Its pdf is as follows.

$$f_Z(z) = \prod_{i=1}^{n} f_{Z_i}(z_i) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} z_i^2 \right) = \left( \frac{1}{\sqrt{2\pi}} \right)^n \exp \left( -\frac{1}{2} \sum_{i=1}^{n} z_i^2 \right) = \left( \frac{1}{\sqrt{2\pi}} \right)^n \exp \left( -\frac{1}{2} z' z \right)$$

This is called standard multivariate normal distribution. Note here that,

$$E[Z] = \begin{pmatrix} E[Z_1] \\ \vdots \\ E[Z_n] \end{pmatrix} = 0_{n \times 1}$$

$$Var(Z) = \begin{pmatrix} E[Z_1 Z_1] & \cdots & E[Z_1 Z_n] \\ \vdots & \ddots & \vdots \\ E[Z_n Z_1] & \cdots & E[Z_n Z_n] \end{pmatrix} = I_n$$

What is the mgf of $Z$? Since $Z_i$’s are independent,

$$M_Z(t) = \prod_{i=1}^{n} M_{Z_i}(t_i) = \prod_{i=1}^{n} \exp \left( \frac{1}{2} t_i^2 \right) = \exp \left( \frac{1}{2} \sum_{i=1}^{n} t_i^2 \right) = \exp \left( \frac{1}{2} t' t \right)$$

Let $X$ have mean $\mu$ and variance $\Sigma$. $X$ has multivariate normal distribution if $X = \Sigma^{1/2} Z + \mu$. Note

$$E[X] = E \left[ \Sigma^{1/2} Z + \mu \right] = \mu$$

$$Var(X) = Var \left( \Sigma^{1/2} Z + \mu \right) = \Sigma^{1/2} Var(Z) \Sigma^{1/2} = \Sigma$$

Since $Z = \Sigma^{-1/2}(X - \mu)$, the Jacobian is

$$J = \det \left( \frac{dZ}{dX} \right) = \det \left( \Sigma^{-1/2} \right) = |\Sigma|^{-\frac{1}{2}}$$

So the pdf is

$$f_X(x) = f_Z \left( \Sigma^{-1/2}(x - \mu) \right) |J|$$

$$= \left( \frac{1}{2\pi} \right)^{\frac{n}{2}} |\Sigma|^{-\frac{1}{2}} \exp \left( -\frac{1}{2} \left[ \Sigma^{-\frac{1}{2}}(x - \mu) \right]' \left[ \Sigma^{-\frac{1}{2}}(x - \mu) \right] \right)$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}} |\Sigma|^{\frac{1}{2}}} \exp \left\{ -\frac{1}{2} (x - \mu)' \Sigma^{-1} (x - \mu) \right\}$$

\(^1\text{Since } \Sigma \text{ is positive definite, its determinant } |\Sigma| \text{ is always positive.}\)
The mgf is

\[ M_X(t) = E[e^{t'X}] = E[e^{t'(\Sigma^{1/2}Z + \mu)}] = e^{t'\mu} E[e^{t'(\Sigma^{1/2}Z)}] = e^{t'\mu} M_Z(\Sigma^{1/2}t) = e^{t'\mu} \exp\left\{ \frac{1}{2} \left( \Sigma^{1/2}t \right)' \left( \Sigma^{1/2}t \right) \right\} = \exp\left( t'\mu + \frac{1}{2} t'\Sigma t \right) \]

2. Problems

See solution file for questions in problem set 4.

2006 mid Q7. Show \( X \) and \( Y - E[Y|X] \) are uncorrelated, in other words, \( \text{Cov}(X, Y - E[Y|X]) = 0 \).

Solution

Denote \( \mu_X = E[X] \) and \( \mu_Y = E[Y] \). Then

\[ E[Y - E[Y|X]] = E[Y] - E[E[Y|X]] = 0 \text{ by LIIE (Law of Iterated Expectations).} \]

So

\[ \text{Cov}(X, Y - E[Y|X]) = E[X(Y - E[Y|X])] - \mu_X \cdot 0 \]

\[ = E[XY] - E[E[XY|X]] \]

\[ = E[XY] - E[E[XY|X]] \]

\[ = 0 \quad \text{(: LIIE)} \]

2007 spring Q1. Let \( X \sim N(0, 1) \). Determine the support and the pdf of \( Y = X^2 \).

Solution

Since \( S_X = \mathbb{R} \), the support of \( Y \) is \( S_Y = \{y|y > 0\} \).

\[ F_Y(y) = \Pr(Y \leq y) = \Pr(X^2 \leq y) \]

\[ = \Pr(-\sqrt{y} \leq X \leq \sqrt{y}) \]

\[ = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} \exp\left( -\frac{1}{2} x^2 \right) dx \]

\[ = 2 \int_{0}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} \exp\left( -\frac{1}{2} x^2 \right) dx \]

Since the pdf is symmetric. Differentiate both side with respect to \( y \), then

\[ f_Y(y) = \frac{2}{\sqrt{2\pi}} \exp\left( -\frac{1}{2} \sqrt{y}^2 \right) \cdot \frac{d}{dy} \frac{\sqrt{y}}{\sqrt{2\pi}} \]

\[ = \frac{1}{\sqrt{\pi} \sqrt{2} y^{1/2} e^{-y/2}} \]

\[ = \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-y/\beta} \]

with \( \alpha = \frac{1}{2} \) and \( \beta = 2 \). So \( Y \) has a \( \Gamma\left( \frac{1}{2}, 2 \right) \) distribution, or \( \chi_1^2 \).
2004 fall Q4. Let the mgf of $X$ be $M(t) = e^{\mu(e^t - 1)}$. Show $E[X] = Var(X)$.

Solution

$$E[X] = \frac{dM(t)}{dt} \Big|_{t=0} = \mu e^{\mu(e^t - 1)} \Big|_{t=0} = \mu$$

$$E[X^2] = \frac{d^2M(t)}{dt^2} \Big|_{t=0} = \left[ \mu e^{\mu(e^t - 1)} + (\mu e^t)^2 e^{\mu(e^t - 1)} \right] \Big|_{t=0} = \mu + \mu^2$$

$$Var[X] = E[X^2] - (E[X])^2 = \mu$$

HMC 2.2.2.

The support of $Y$ is as follows.

$$S_Y = \{(1, 1), (2, 1), (3, 1), (2, 2), (4, 2), (6, 2), (3, 3), (6, 3), (9, 3)\}$$

Find the joint pmf as $p_Y(4, 2) = p_X(2, 2) = \frac{4}{36}$, and so on.

<table>
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<th>joint pmf</th>
<th>$Y_1$ = 1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>6</th>
<th>9</th>
</tr>
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<tr>
<td>$Y_2 = 1$</td>
<td>$\frac{1}{36}$</td>
<td>$\frac{2}{36}$</td>
<td>$\frac{3}{36}$</td>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>$\frac{2}{36}$</td>
<td>0</td>
<td>$\frac{4}{36}$</td>
<td>$\frac{6}{36}$</td>
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<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>$\frac{3}{36}$</td>
<td>0</td>
<td>$\frac{6}{36}$</td>
<td>$\frac{9}{36}$</td>
</tr>
</tbody>
</table>

marg. pmf of $Y_1$

| $Y_1$ | $\frac{1}{36}$ | $\frac{4}{36}$ | $\frac{6}{36}$ | $\frac{4}{36}$ | $\frac{12}{36}$ | $\frac{9}{36}$ |

HMC 2.4.3.

Note that marginal pdf’s are

$$f_X(x) = \int_x^1 f(x, y)dy = 2(1 - x)$$

$$f_Y(y) = \int_0^y f(x, y)dy = 2y$$

So the conditional expectations are

$$E[Y | X = x] = \int_x^1 y f_Y | X = x(y)dy = \int_x^1 y f(x, y)f_X(x)dy = \int_x^1 \frac{y}{1 - x}dy = \frac{1 + x}{2}$$

$$E[X | Y = y] = \int_0^y x f_X | Y = x(x)dx = \int_0^y x f(x, y)f_Y(y)dx = \int_0^y \frac{x}{y}dy = \frac{y}{2}$$

$Cov(X, Y)$ is obtained by

$$E[XY] = \int_0^1 \int_x^1 xy f(x, y)dydx = \int_0^1 \int_x^1 x(1 - x^2)dx = \frac{1}{4}$$

$$E[X] = E[E[X|Y]] = \int_0^1 \frac{y}{2} f_Y(y)dy = \frac{1}{3}$$

$$E[Y] = E[E[Y|X]] = \int_0^1 \frac{x + 1}{2} f_X(x)dx = \frac{2}{3}$$

$$Cov(X, Y) = E[XY] - E[X]E[Y] = \frac{1}{36}$$
Also
\[ \sigma_X^2 = \int_0^1 \int_x^1 x^2 f(x,y) \, dy \, dx - (E[X])^2 = \frac{1}{6} - \frac{1}{9} = \frac{1}{18} \]
\[ \sigma_Y^2 = \int_0^1 \int_x^1 y^2 f(x,y) \, dy \, dx - (E[Y])^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18} \]

Hence
\[ \rho_{XY} = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{1}{2} \]

HMC 2.5.8.
Even if the joint pdf can be factorized, they are not independent since the support is not rectangular!
In fact,
\[ f_X(x) = \int_0^x f(x,y) \, dy = 3x^2 \]
\[ f_Y(y) = \int_y^1 f(x,y) \, dx = \frac{3}{2} (1 - y^2) \]
So \( f(x,y) \neq f_X(x) f_Y(y) \).
\[ E[X|Y = y] = \int_y^1 x f_{X|Y=y}(x) \, dx = \int_y^1 x f(x,y) f_Y(y) \, dx = \int_y^1 \frac{2x^2}{1-y^2} \, dy = \frac{2(1-y^3)}{3(1-y^2)} \]

HMC 2.5.13.
Since \( X_1 \) and \( X_2 \) are independent, the mgf of \( Y \) is
\[ M_Y(t) = E \left[ e^{t(X_1+X_2)} \right] = E \left[ e^{tX_1} \right] E \left[ e^{tX_2} \right] = M_{X_1}(t) M_{X_2}(t) = \frac{e^{2t}}{(2-e^t)^2} \]
So the mean and the variance of \( Y \) are
\[ E[Y] = \left. \frac{dM(t)}{dt} \right|_{t=0} = \left[ \frac{2e^{2t}}{(2-e^t)^2} + (-2) \cdot \frac{e^{2t}}{(2-e^t)^3} \cdot (-e^t) \right]_{t=0} = \left. \frac{4e^{2t}}{(2-e^t)^3} \right|_{t=0} = 4 \]
\[ E[Y^2] = \left. \frac{d^2M(t)}{dt^2} \right|_{t=0} = \left[ \frac{8e^{2t}}{(2-e^t)^3} + (-3) \cdot \frac{4e^{2t}}{(2-e^t)^4} \cdot (-e^t) \right]_{t=0} = 20 \]
\[ \text{Var}[Y] = E[Y^2] - (E[Y])^2 = 4 \]

2006 fall Q4 / 2003 fall Q3. Let \( X_i \sim N(\mu_i, \sigma_i^2) \) independently distributed. Show that for any scalars \( a_i \)'s, \( \sum_{i=1}^n a_i X_i \sim N(\sum_{i=1}^n a_i \mu_i, \sum_{i=1}^n a_i^2 \sigma_i^2) \).
Hint: Use mgf.

2005 fall Q3. Let \( X_i \) Poisson(\( \mu_i \)) independently distributed. Show \( \sum_{i=1}^n X_i \) Poisson(\( \sum_{i=1}^n \mu_i \)).
Hint: Use mgf.