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Problem Set VI (due Wed 5/20/09)

1. Let $\{y_t\}$ be a stationary AR(2) process generated by the difference equation

$$(1-\alpha B)(1-\beta B)y_t = \varepsilon_t,$$

where the $\{\varepsilon_t\}$ are white noise. Assume $\alpha \neq \beta$ and both parameters are less than one in absolute value. Find the coefficients c_j such that y_t has the representation

$$y_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}.$$

2. (a) Given two observations y_1 and y_2 from the causal AR(1) process

$$Y_t - \alpha Y_{t-1} = U_t,\ \{U_t\} \sim \text{iid}N(0,\sigma^2),$$

such that $|y_1| \neq |y_2|$, find the maximum likelihood estimates of α and σ^2.

(b) Let $\{X_t\}$ be a stationary process with mean μ. Show that the optimal linear forecast of X_{n+h} based on $1, X_1, ..., X_n$ equals μ plus the optimal linear forecast of Y_{n+h} based on $Y_1, ..., Y_n$, where the zero mean process $\{Y_t\}$ is defined by $Y_t := X_t - \mu$ and where h is a fixed integer ≥ 1.

3. Assume the model is given by

$$\begin{aligned} Y_t &= \mu + \alpha Y_{t-1} + U_t \text{ for } t = 1, ..., T, \\ U_t &\equiv iid\ N(0,\sigma^2) \text{ for } \sigma^2 > 0, \\ Y_0 &\sim N(\mu, \sigma^2/(1-\alpha^2)). \end{aligned}$$

For $T = 100$, $\mu = .5$, $\alpha = 0, .3, .6, .9, .99$ and $\sigma^2 = 1$ generate $R = 2000$ time series according to this model. Each time, calculate the OLS estimator $\widehat{\alpha}$ of α (from a regression of Y_t on a constant and Y_{t-1}). For the five simulation designs, report mean and median bias, standard deviation and RMSE (root mean squared error) of $\widehat{\alpha}$. Discuss your findings on the finite-sample behavior of the OLS estimator. Calculate nominal 95% confidence intervals (CI) for α obtained from inverting a t–test. Report the empirical coverage probability of the CI. Discuss your findings.

4. Suppose that an MA(2) model is estimated when the second moving average parameter, θ_2, is actually equal to zero. Find an expression for the relative efficiency of the resulting estimator of θ_1 as compared with the estimator obtained from an MA(1) model.

Hints:

Equation (5.5.4) in Hamilton for the conditional likelihood of an MA(q) process (conditioned on $\varepsilon_0 = 0, ..., \varepsilon_{-q+1} = 0$, assuming $\mu = 0$ for simplicity) reads

$$\begin{aligned}
\mathcal{L} &= const - \frac{T}{2}\log\sigma^2 - \frac{1}{2\sigma^2}S(\theta), \text{ for} \\
S(\theta) &: = \textstyle\sum_{t=1}^T \varepsilon(\theta)_t^2, \\
\varepsilon(\theta)_t &: = y_t - \theta_1\varepsilon_{t-1} - ... - \theta_q\varepsilon_{t-q}, \\
\theta &: = (\theta_1, ..., \theta_q).
\end{aligned} \tag{1}$$

Leaving out the argument θ in $\varepsilon(\theta)_t$ (to simplify notation) we have

$$\begin{aligned}
\partial\mathcal{L}/\partial\theta &= -\frac{1}{\sigma^2}\textstyle\sum_{t=1}^T(\partial\varepsilon_t/\partial\theta)\varepsilon_t = \frac{1}{\sigma^2}\sum_{t=1}^T x_t\varepsilon_t \in R^{1\times q}, \text{ where} \\
x_t &: = -(\partial\varepsilon_t/\partial\theta).
\end{aligned}$$

From ML theory we know how to express the asymptotic variance of the ML estimators in terms of the information matrix

$$\begin{aligned}
E(\partial\mathcal{L}/\partial\theta)'(\partial\mathcal{L}/\partial\theta) &= E(\frac{1}{\sigma^2}\textstyle\sum_{t=1}^T x_t\varepsilon_t)'\frac{1}{\sigma^2}\sum_{s=1}^T x_s\varepsilon_s \\
&= \frac{1}{\sigma^4}\textstyle\sum_{t=1}^T Ex_t'x_tE\varepsilon_t^2 \\
&= E\frac{1}{\sigma^2}\textstyle\sum_{t=1}^T x_t'x_t.
\end{aligned}$$

In fact, the asymptotic variance of the MLE estimator is the limit of T-times the inverse of this matrix, $\lim_{T\to\infty}T(E\frac{1}{\sigma^2}\sum_{t=1}^T x_t'x_t)^{-1}$ or the inverse of the probability limit of $\frac{1}{\sigma^2}\frac{1}{T}\sum_{t=1}^T x_t'x_t$.

The case of MA(1):

By (1), we have

$$(\partial\varepsilon_t/\partial\theta) = -\theta_1(\partial\varepsilon_{t-1}/\partial\theta) - \varepsilon_{t-1}$$

and thus it follows that x_t follows an AR(1):

$$x_t = -\theta_1 x_{t-1} + \varepsilon_{t-1}.$$

Therefore

$$p\lim T^{-1}\textstyle\sum_{t=1}^T x_t x_t' = var x_t = \sigma^2/(1-\theta_1^2),$$

the formula for the variance of an AR(1). Therefore, using the above information matrix formula, it follows that the asymptotic variance of the conditional MLE of θ_1 is given by

$$(1-\theta_1^2).$$

Generalize this approach to MA(2) and you are done.